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On Stable Difference Scheme for Identification Elliptic Problem with Integral and Second Kind Boundary Condition

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Abstract. In this study, the first order of accuracy difference scheme for approximate solution of source identification elliptic problem with integral and second kind boundary conditions is considered. Stability and coercive stability estimates for solution of the difference scheme are described. Moreover, test example with computation results is given.

Keywords: Difference schemes, elliptic problem, source identification, stability.
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INTRODUCTION

The theory of inverse problems for differential, differential-functional and difference equations is intensively developed scientific area of modeling various real life processing. The theory and methods of solving inverse problems to identify the unknown parameters of equation have been comprehensively studied by several researchers (see [1–21] and references therein).

It is well known that in studying direct problems, the solution of known differential equation or differential scheme is obtained by using the initial and boundary conditions or their corresponding approximations, while in the inverse problems, part of equation or coefficients are also unknowns. To determine the governing equation it is required to satisfy some additional conditions in comparison with the corresponding direct problem. In paper [20], the authors gave review results on approximation of inverse problems for different types of partial differential equations in functional spaces and presented general statements.

In work [5], the author studied inverse problem for an elliptic differential equation with Neumann conditions and established stability and coercive stability estimates for the solution of inverse problem. The first and second order of accuracy stable difference schemes were presented. Papers [6, 7] are devoted to getting various estimates for the solution of source identification problem and well-posedness of difference schemes for elliptic problem with Neumann type overdetermination. In study [10], stability estimates for solutions of Neumann-type elliptic overdetermined multidimensional differential problems with integral condition were established.

In this study, we propose the first order of accuracy difference scheme for approximate solution of source identification elliptic problem with integral and second kind boundary conditions. Stability and coercive stability estimates for solution of difference problem are described. By using MATLAB program, test example with computation results is given.

MAIN RESULTS

Difference Scheme and Stability Estimates

Let $\Omega = (0, L)^n$ be open cube in the n-dimensional vector space R_n , $\overline{\Omega} = \Omega \cup \partial\Omega$ and

$$a_1(x), ..., a_n(x), \varphi(x), \psi(x), \xi(x), \ (x \in \Omega), \ f(t, x) \ (x \in \Omega, t \in (0, T) \ )$$

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be given sufficiently smooth functions. In addition, $a_s(x) \geq \delta > 0, s = 1, ..., n, \forall\ x \in \Omega$.

In $\Omega \times [0, T]$, we consider the following identification multidimensional elliptic problem with integral and second kind boundary conditions:

$$\begin{cases} -v_{tt}(x,t) - \sum\limits_{r=1}^{n}(a_r(x)v_{x_r}(x,t))_{x_r} + \sigma v(x,t) = f(x,t) + p(x),\ x \in \Omega, t \in (0,T), \\ v(x,0) = \varphi(x),\ v(x,T) = \int\limits_0^T \mu(\lambda)\, v(\lambda, x)\, d\lambda + \zeta(x),\ v(\gamma, x) = \eta(x), x \in \overline{\Omega}\ (0 < \gamma < T), \\ \frac{\partial}{\partial \vec{n}} v(x,t) = 0,\ x \in S = \partial\Omega,\ t \in [0,T]. \end{cases} \tag{1}$$

We denote

$$\widetilde{\Omega}_h = \{x_m = (h_1 m_1, \cdots, h_n m_n); m = (m_1, \cdots, m_n), m_r = 0, \cdots, N_r,\ h_r N_r = 1, r = 1, \cdots, n\},$$

$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap \partial\Omega.$$

Let A_h^x be the operator $A_h^x u^h(x) = -\sum_{r=1}^n \left(a_r(x) u^h_{\bar{x}_r}(x)\right)_{x_r, j_r} + \sigma u^h(x)$ acting in the space of corresponding grid functions $u^h(x)$ such that $u^h(x) = 0$ on $x \in S_h$. To formulate stability results, we denote by
$L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ functional spaces of the grid functions $\rho^h(x) = \{\rho(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$ and equipped with following norms, respectively

$$]\, \|\rho\|_{L_{2h}} = \left(\sum_{x \in \widetilde{\Omega}_h} |\rho^h(x)|^2 h_1 \cdots h_n \right)^{1/2},$$

$$\left\|\rho^h\right\|_{W_{2h}^2} = \left\|\rho^h\right\|_{L_{2h}} + \left(\sum_{x \in \widetilde{\Omega}_h} \sum_{s=1}^n \left|(\rho^h(x))_{x_s \overline{x_s}, n_s}\right|^2 h_1 \cdots h_n \right)^{1/2}.$$

By using notation A_h^x, problem (1) is replaced by the the first order of accuracy difference scheme

$$\begin{cases} -\tau^{-2}\left[v_{k+1}^h(x) - 2v_k^h(x) + v_{k-1}^h(x)\right] + A_h^x v_k^h(x) = p^h(x) + f_k^h(x), 1 \leq k \leq N-1,\ x \in \Omega_h, \\ v_N^h(x) = \sum\limits_{i=0}^{N-1} \tau\rho(t_i) v_i^h(x) +\ \eta^h(x), v_0^h(x) = \varphi^h(x), v_l^h(x) = \zeta^h(x)\ , x \in \widetilde{\Omega}_h. \end{cases} \tag{2}$$

Theorem 1. For the solution of difference scheme (2), the stability estimate

$$\left\|\left\{v_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})} \leq M(\delta, \mu)\left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\zeta^h\right\|_{L_{2h}} + \left\|\ \eta^h\right\|_{L_{2h}} + \left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})}\right],$$

is valid, where $M(\delta, \mu)$ is independent of $\tau, \varphi^h, \zeta^h, \eta^h$ and $\left\{f_k^h\right\}_1^{N-1}$.

Theorem 2. Solution of difference scheme (2) satisfies the following coercive stability estimate:

$$\left\|\left\{\tau^{-1}(v_k^h - v_{k-1}^h)\right\}_1^{N-1}\right\|_{C_1^\alpha(L_{2h})}$$

$$\leq M(\delta, \mu, \alpha)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\ \eta^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C_1^\alpha(W_{2h}^2)}\right],$$

where $M(\delta, \mu, \alpha)$ is independent of $\tau, \left\{f_k^h\right\}_1^{N-1}, \varphi^h, \zeta^h, \eta^h$.

Numerical Test Example

Now, we will layout numerical results for 2D source identifying elliptic problem with integral condition. Presented results are carried out by using MATLAB. In $[0, 1] \times [0, 1]$, we consider the following source identification problem for 2D elliptic partial differential equation with integral and second kind boundary conditions:

$$\begin{cases} -v_{tt}(t,x) - (1+x^2)v_{xx}(t,x) - 2xv_x(t,x) + v(t,x) = f(t,x) \; + \; p(x), 0 < x < 1, \; 0 < t < 1, \\ v(0,x) = \varphi(x), \; v(0.3,x) = \zeta(x), v(1,x) = \int\limits_0^1 e^{-\lambda} v(\lambda, x) d\lambda \; + \eta(x), \; 0 \le x \le 1, \\ v_x(t,0) = 0, \; v_x(t,1) = 0, \; 0 \le t \le 1. \end{cases} \tag{3}$$

Here,

$$f(t,x) = \pi^2(1+x^2)e^{-t} \; \cos(\pi x) + 2x\pi e^{-t} \; \sin(\pi x) \;, \varphi(x) = \cos(\pi x),$$

$$\eta(x) = \left(-\frac{1}{2} + \frac{e^{-2}}{2} + 2e^{-1}\right)\cos(\pi x), \zeta(x) = \left(e^{-0.3} \; + 1\right)\cos(\pi x).$$

Let

$$N\tau = 1, t_k = k\tau, k = 0, ..., N, Mh = 1, x_n = nh, n = 0, ...M, l = [0.3\tau] \,, \omega_0 = 0.3\tau - l,$$
$$\varphi_n = \varphi\left(x_n\right), \psi_n = \psi\left(x_n\right), \zeta_n = \zeta\left(x_n\right); f_n^k = f(t_k, x_n), k = 0, ..., N, \; n = 0, ...M.$$

Applying difference scheme (2), we get the first order of accuracy difference scheme for approximation of problem (3)

$$\begin{cases} -\frac{1}{\tau^2}\left(v_n^{k+1} - 2v_n^k + v_n^{k-1}\right) - \frac{\left(1+x_n^2\right)}{h^2}\left(v_{n+1}^k - 2v_n^k + v_{n-1}^k\right) \\ -\frac{1}{2h}\left(v_{n+1}^k - v_{n-1}^k\right) + v_n^k = \; f(t_k, x_n) + p_n, \; n = 1, ...M-1, k = 1, ...N-1, \\ v_0^k = 0, v_M^k = 0, \; k = 0, ..., N, \\ v_n^0 = \varphi_n, v_n^l = \zeta_n, v_n^N - \sum\limits_{j=0}^{N-1} \tau e^{-t_j} \; v_n^j = \eta_n, \; n = 0, ...M. \end{cases} \tag{4}$$

Firstly, we find solution of the following difference scheme

$$\begin{cases} \frac{1}{\tau^2}\left(u_n^{k+1} - 2u_n^k + u_n^{k-1}\right) + \frac{\left(1+x_n^2\right)}{h^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right) \\ +\frac{x_n}{h}\left(u_{n+1}^k - u_{n-1}^k\right) - u_n^k = -f(t_k, x_n), \; n = \overline{1, M-1}, k = \overline{1, N-1}, \\ u_0^k = 0, u_M^k = 0, \; k = 0, \cdots, N, \\ u_n^0 - u_n^l = \varphi_n - \zeta_n, u_n^N - \mu u_n^0 - \sum\limits_{j=0}^{N-1} \tau e^{-t_j} \; u_n^j = -\mu\varphi_n + \eta_n, \; n = \overline{0, M}. \end{cases} \tag{5}$$

Secondly, we calculate p_n. It is defined by

$$\begin{aligned} p_n = &- \frac{\left[(\zeta_{n+1} - u_{n+1}^l) - 2(\zeta_n - u_n^l) + (\zeta_{n-1} - u_{n-1}^l)\right](1+x_n^2)}{h^2} \\ &- \frac{x_n(\zeta_{n+1} - u_{n+1}^l - \zeta_{n-1} + u_{n-1}^l)}{2h} + (\zeta_n - u_n^l), \quad n = \overline{1, M-1}. \end{aligned} \tag{6}$$

Difference problem (5) can be rewritten in the matrix form

$$\begin{aligned} &A_n u_{n+1} + B_n u_n + C_n u_{n-1} = I g^{(n)}, \; n = \overline{1, M-1}, \\ &u_0 = u_1, u_M = u_{M-1}. \end{aligned} \tag{7}$$

Here, $g^{(n)}$ is an $(N+1)\times 1$ column matrix, I is an $(N+1)\times(N+1)$ identity matrix, u_s is the $(N+1)\times 1$ matrix $u_s = \begin{bmatrix} u_s^0 & u_s^1 & \dots & u_s^{N-1} & u_s^N \end{bmatrix}^t$, $s = n-1, n, n+1$, and A_n, B_n, C_n are square matrices,

$$
\begin{aligned}
&A_n = diag(0, a_n, a_n, \dots, a_n, 0), C_n = diag(0, c_n, c_n, \dots, c_n, 0),\\
&a_n = \frac{\left(1+x_n^2\right)}{h^2} + \frac{x_n}{2h}, c_n = \frac{\left(1+x_n^2\right)}{h^2} - \frac{x_n}{2h}, q = -\frac{2\left(1+x_n^2\right)}{h^2} - \frac{2}{\tau^2} - 1, r = \frac{1}{\tau^2}, \mu = 1 - \sum_{i=0}^{N-1} e^{-t_i}\tau,\\
&g_k^{(n)} = -f(x_n, t_k),\ k = \overline{1, N-1},\ n = \overline{1, M-1}, g_n^0 = \varphi_n - \zeta_n,\ g_n^N = -\mu\varphi_n + \eta_n,\ n = 1, ..., M-1,\\
&b_{i,i} = q, b_{i-1,i} = r, b_{i,i-1} = r, i = 2, ..., N, b_{1,1} = 1, b_{1,l+1} = -1, b_{N+1,N+1} = 1,\\
&b_{N+1,N} = -\tau e^{-t_{N-1}}, b_{N+1,1} = -\mu - e^{-t_0}\tau, b_{N+1,j} = -\tau e^{-t_j}, j = 2, ..., N-1,\\
&b_{i,j} = 0, for\ other\ i\ and\ j.
\end{aligned}
$$

We will apply modified Gauss elimination method to calculate numerical solution of (7) and seek a solution by the following form

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1},\ n = M-1, ..., 1, 0, \tag{8}$$

where α_n $(1 \le n \le M-1)$ are square matrices and β_n $(1 \le n \le M-1)$ are column vectors, calculated as,

$$\alpha_{n+1} = -(B_n + C_n\alpha_n)^{-1}A_n,\ \beta_{n+1} = (B_n + C_n\alpha_n)^{-1}(RF_n - C_n\beta_n),\ n = 1, 2, ..., M-1, \tag{9}$$

α_1 is an identity matrix and β_1 column vector with $(N+1)$ zero elements,

$$u_M = u_{M-1} = (A_M + B_M + C_M\alpha_{M-1})^{-1}\left(F_M - C_M\beta_{M-1}\right).$$

Finally, in the third stage, we define $\left\{u_n^k\right\}$ by

$$v_n^k = u_n^k + \zeta_n - u_n^l. \tag{10}$$

Errors are presented in Tables I,II,III in case $(N, M) = (10, 10)$, $(N, M) = (20, 20)$, $(N, M) = (40, 40)$, $(N, M) = (80, 80)$ and $(N, M) = (160, 160)$, $(N, M) = (320, 320)$, correspondingly.

TABLE 1. Errors for u

DS/(N, M)	(10,10)	(20,20)	(40,40)	(80,80)	(160,160)	(320,320)
(5)	0.3873	0.1853	0.0906	0.0448	0.0223	0.0111

TABLE 2. Errors for v

DS/(N, M)	(10,10)	(20,20)	(40,40)	(80,80)	(160,160)	(320,320)
(4)	0.2349	0.1243	0.0639	0.0324	0.0163	0.0082

TABLE 3. Errors for p

DS/(N, M)	(10,10)	(20,20)	(40,40)	(80,80)	(160,160)	(320,320)
(4)	0.0745	0.0340	0.0163	0.0080	0.0039	0.0019

CONCLUSION

In this work, the first order of accuracy difference scheme for approximate solution of source identification elliptic problem with integral and second kind boundary conditions is discussed. Stability and coercive stability estimates for solution of difference scheme are described. Finally, test example with computation results is given.

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