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A finite difference method for the inverse elliptic problem with the Dirichlet condition

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Abstract. Well-posedness of the inverse problem for the elliptic differential equation with Dirichlet condition is investigated. A finite difference method for the approximate solution of the inverse problem is applied. Stability and coercive stability estimates for the solution of the first and second order of accuracy difference schemes are obtained. In applications, the inverse problem for the multidimensional elliptic equation is studied. The theoretical statements are supported by the numerical example in a two dimensional case of elliptic equation.

Key words. Difference scheme, inverse elliptic problem, well-posedness, stability, coercive stability.

1 Introduction

We consider the inverse problem of finding a function u and a element p for the elliptic equation

$$\begin{cases} -u_{tt}(t)+Au(t)=f(t)+p, 0<t<T, \\ u(0)=\varphi,\ u(T)=\psi, u(\lambda)=\xi, 0<\lambda<T \end{cases} \tag{1.1}$$

in an arbitrary Hilbert space H with the self-adjoint positive definite operator A. Here, φ, ψ, and ξ are given elements of H, λ is known number.

It is clear that for finding a solution $u(t)$ of problem (1.1) it is useful to apply the substitution

$$u(t)=v(t)+A^{-1}p, \tag{1.2}$$

where $v(t)$ is the solution of the following nonlocal boundary value problem

$$\begin{cases} -v_{tt}(t) + Av(t) = f(t), 0 < t < T, \\ v(0) - v(\lambda) = \varphi - \xi, \ v(T) - v(\lambda) = \psi - \xi, \end{cases} \tag{1.3}$$

and p is the unknown element defined by formulas

$$p = A\varphi - Av(0) \ \text{ or } p = A\psi - Av(T). \tag{1.4}$$

So, we consider the algorithm which includes three stages for solving problem (1.1). In the first stage, we consider nonlocal boundary value problem (1.3) and we will obtain $v(t)$.

In the second stage, we will put $t = 0$ or $t = T$, and find $v(0)$ or $v(T)$. Then, using (1.4), we will obtain p. In the third stage, we will use formula (1.2) for obtaining the solution $u(t)$ of problem (1.1). Moreover, we have one more possibility. Actually, we can obtain $u(t)$ by formula $u(t) = v(t) + w(t)$, where $w(t)$ is the solution of the boundary value problem

$$\begin{cases} -w_{tt}(t) + Aw(t) = p, 0 < t < T, \\ w(0) = \xi - v(\lambda), \ w(T) = \xi - v(\lambda). \end{cases} \tag{1.5}$$

Inverse problems play an important role on mathematical modeling of real processes (see, for example [1–3] and the references therein). Well-posedness of inverse problems for partial differential equations have been studied extensively by many researchers (see, e.g. [4–16]). Various inverse problems for elliptic type equations can be reduced to the nonlocal boundary-value problems. Methods of the solution of nonlocal boundary values problems of various differential and difference equations of elliptic type have been investigated extensively by many researchers (see [17–26] and the references therein).

In [5], existence and uniqueness theorems for problem (1.1) in a Banach space were presented. However, stability estimates for solution of (1.1) were not established. In the present paper, we establish stability and coercive stability estimates for solution of inverse problem (1.1). Moreover, the first and second order of accuracy difference schemes for the approximate solution of problem (1.1) are presented. Stability and coercive stability estimates for the solution of these difference schemes are established. In applications, the inverse problem for the multidimensional elliptic equation with Dirichlet condition

$$\begin{cases} -u_{tt}(t,x)-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}=f(t,x)+p(x), \\ x=(x_1,\cdots,x_n)\in\Omega,\ 0<t<T, \\ u(0,x)=\varphi(x), u(T,x)=\psi(x), u(\lambda,x)=\xi(x), x\in\overline{\Omega}, \\ u(t,x)=0,\ x\in S,\ 0\leq t\leq T \end{cases} \tag{1.6}$$

is studied. Here, $a_r(x)$ $(x\in\Omega)$, $\varphi(x),\psi(x),\xi(x)$ $(x\in\overline{\Omega})$, and $f(t,x)$ $(t\in(0,T),\ x\in\Omega)$ are given smooth functions and $a_r(x)\geq a>0$ $(x\in\Omega)$, and $\Omega=(0,\ell)\times\cdots\times(0,\ell)$ is the open cube in the n-dimensional Euclidean space with boundary S, $\overline{\Omega}=\Omega\cup S$.

In [16] inverse problem for the multidimensional elliptic equation with Neumann condition is investigated. In the present work, the first and second order of accuracy in t and second order of accuracy in space variables for the approximate solution of problem (1.6) are presented. The stability and coercive stability estimates for the solution of these difference schemes are obtained. The modified Gauss elimination method is used for solving these difference schemes.

2 Well-posedness of problem (1.1) and its applications

Theorem 2.1 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $f(t)\in\mathcal{C}_{0T}^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $(u(t),p)$ of problem (1.1) the following stability estimates hold:*

$$\|u\|_{C(H)}\leq M\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\|f\|_{C(H)}\right], \tag{2.1}$$

$$\left\|A^{-1}p\right\|_H\leq M\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\|f\|_{C(H)}\right], \tag{2.2}$$

$$\|p\|_H\leq M\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\frac{1}{\alpha(1-\alpha)}\|f\|_{\mathcal{C}_{0T}^{\alpha,\alpha}(H)}\right], \tag{2.3}$$

where M does not depend on α, φ,ψ,ξ, and $f(t)$.

Here, $\mathcal{C}_{0T}^{\alpha,\alpha}(H)$ is the space obtained by completion of the space of all smooth H-valued functions ρ on $[0,T]$ with the norm

$$\|\rho\|_{\mathcal{C}_{0T}^{\alpha,\alpha}(H)}=\|\rho\|_{\mathcal{C}(H)}+\sup_{0\leq t<t+\tau\leq T}\frac{(t+\tau)^\alpha(T-t)^\alpha\|\rho(t+\tau)-\rho(t)\|_H}{\tau^\alpha}.$$

Proof. We will obtain the representation formula for solution of auxiliary problem (1.3). Applying formula of [27, p.205], we get

$$\begin{aligned} v(t) &= (I-e^{-2TB})^{-1}\{(e^{-tB}-e^{-(2T-t)B})v(0) \\ &+(e^{-(T-t)B}-e^{-(T+t)B})v(T)-(e^{-(T-t)B}-e^{-(T+t)B}) \\ &\times(2B)^{-1}\int_0^T(e^{-(T-s)B}-e^{-(T+s)B})f(s)ds\} \\ &+(2B)^{-1}\int_0^T(e^{-|t-s|B}-e^{-(t+s)B})f(s)ds. \end{aligned} \tag{2.4}$$

Here, $B=A^{\frac{1}{2}}$. Using formula (2.4) and nonlocal boundary conditions, we get the system of equations

$$\begin{cases} v(0)=(I-e^{-2TB})^{-1}[(e^{-\lambda B}-e^{-(2T-\lambda)B})v(0) \\ +(e^{-(T-\lambda)B}-e^{-(T+\lambda)B})v(T)]-(I-e^{-2TB})^{-1} \\ \times(e^{-(T-\lambda)B}-e^{-(T+\lambda)B})(2B)^{-1}\int\limits_0^T(e^{-(T-s)B}-e^{-(T+s)B})f(s)ds \\ +(2B)^{-1}\int\limits_0^T(e^{-|\lambda-s|B}-e^{-(\lambda+s)B})f(s)ds+\varphi-\xi, \\ v(T)=v(0)+\psi-\varphi. \end{cases}$$

Solving this system, we get

$$\begin{aligned} v(0) &= -(I-e^{-\lambda B})^{-1}(I-e^{-(T-\lambda)B})^{-1}(I+e^{-TB}) \\ &\times(e^{-(T-\lambda)B}+e^{-TB})(\psi-\varphi+(2B)^{-1} \\ &\times\int_0^T(e^{-(T-s)B}-e^{-(T+s)B})f(s)ds) \\ &+(I-e^{-\lambda B})^{-1}(I-e^{-(T-\lambda)B})^{-1}(I+e^{-TB}) \\ &\times(\varphi-\xi+(2B)^{-1}\int_0^T\left(e^{-|\lambda-s|B}-e^{-(\lambda+s)B}\right)f(s)ds), \end{aligned} \tag{2.5}$$

$$v(T) = v(0)+\psi-\varphi. \tag{2.6}$$

So, problem (1.3) has a unique solution $v(t)$ that is defined by formulas (2.4), (2.5), and (2.6). Applying formulas (2.4), (2.5), (2.6), and method of monograph [27], we get

$$\|v\|_{C(H)}\leq M\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\|f\|_{C(H)}\right], \tag{2.7}$$

$$\begin{aligned} &\|v''\|_{C_{0T}^{\alpha,\alpha}(H)}+\|Av\|_{C_{0T}^{\alpha,\alpha}(H)} \\ \leq\ & M\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha,\alpha}(H)}\right]. \end{aligned} \tag{2.8}$$

The proofs of estimates (2.2), (2.3) are based on formula (1.2), estimates (2.7) and (2.8). Applying formula (1.4) and estimates (2.7), (2.2), we can get estimate (2.1). Theorem 2.1 is proved. ■

Theorem 2.2 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi \in D(A)$ and $f(t) \in \mathcal{C}_{0T}^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $(u(t),p)$ of problem (1.1) the following coercive inequality holds:*

$$\begin{aligned} & \|u''\|_{\mathcal{C}_{0T}^{\alpha,\alpha}(H)} + \|Au\|_{\mathcal{C}_{0T}^{\alpha,\alpha}(H)} + \|p\|_H \\ \leq \; & M[\frac{1}{\alpha(1-\alpha)}\|f\|_{\mathcal{C}_{0T}^{\alpha,\alpha}(H)} + \|A\varphi\|_H + \|A\psi\|_H + \|A\xi\|_H], \end{aligned} \tag{2.9}$$

where M is independent of α, φ,ψ,ξ, and $f(t)$.

The proof of estimate (2.9) is based on formula (1.2), estimates (2.8) and (2.3).

Now, we consider the application of abstract Theorems 2.1 and 2.2. We consider problem (1.6). It is known that the differential expression ([28])

$$A^x u(x) = -\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}$$

defines a self-adjoint positive definite operator A^x acting on $L_2(\overline{\Omega})$ with the domain $D(A^x) = \left\{u(x) \in W_2^2(\overline{\Omega}),\ u(x)=0 \text{ on } S\right\}$.

Therefore, we can replace inverse problem (1.6) by abstract boundary problem (1.1) in $H = L_2(\overline{\Omega})$. Using the results of Theorems 2.1 and 2.2, we can obtain the following theorem on well-posedness of problem (1.6).

Theorem 2.3 *For the solution of inverse boundary value problem (1.6), the following stability estimate is valid:*

$$\|u\|_{C(L_2(\overline{\Omega}))} \leq M[\|\varphi\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} + \|\xi\|_{L_2(\overline{\Omega})} + \|f\|_{C(L_2(\overline{\Omega}))}],$$

where M is independent of $\varphi(x),\psi(x),\xi(x)$, and $f(t,x)$.

Theorem 2.4 *For the solution of inverse boundary value problem (1.6), the following estimate is valid:*

$$\begin{aligned} & \|u''\|_{\mathcal{C}_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} + \|u\|_{\mathcal{C}_{0T}^{\alpha,\alpha}(W_2^2(\overline{\Omega}))} + \|p\|_{L_2(\overline{\Omega})} \\ & \leq M[\frac{1}{\alpha(1-\alpha)}\|f\|_{\mathcal{C}_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} + \|\varphi\|_{W_2^2(\overline{\Omega})} + \|\psi\|_{W_2^2(\overline{\Omega})} + \|\xi\|_{W_2^2(\overline{\Omega})}], \end{aligned}$$

where M does not depend on $\alpha,\varphi(x),\psi(x),\xi(x)$, and $f(t,x)$.

The proofs of Theorems 2.3 and 2.4 are based on the symmetry properties of the operator A^x in $L_2(\overline{\Omega})$ and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 2.5 ([29]) *For the solution of the elliptic differential problem*

$$\begin{cases} A^x w(x) = \omega(x),\ x \in \Omega, \\ w(x) = 0,\ x \in S, \end{cases}$$

the following coercivity inequality holds :

$$\|w\|_{W_2^2(\overline{\Omega})} \le M \|\omega\|_{L_2(\overline{\Omega})},$$

where M does not depend on $\omega(x)$.

3 Well-posedness of difference schemes for problem (1.1) and its applications

Well-posedness of various types of the difference schemes for elliptic equations had been investigated in [27, 29–31] (see also references therein). Applying the approximate formulas

$$\begin{aligned} u(\lambda) &= u([\frac{\lambda}{\tau}]\tau) + o(\tau), \\ u(\lambda) &= u([\frac{\lambda}{\tau}]\tau) + (\frac{\lambda}{\tau} - [\frac{\lambda}{\tau}])(u([\frac{\lambda}{\tau}]\tau + \tau) - u([\frac{\lambda}{\tau}]\tau)) + o(\tau^2) \end{aligned}$$

for $u(\lambda) = \xi$, inverse problem (1.1) corresponds to the following first and second order of accuracy difference problems

$$\begin{cases} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = \theta_k + p,\ \theta_k = f(t_k), \\ t_k = k\tau,\ 1 \le k \le N-1,\ N\tau = T, u_0 = \varphi, u_N = \psi, u_l = \xi, \end{cases} \tag{3.1}$$

$$\begin{cases} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = \theta_k + p,\ \theta_k = f(t_k), \\ t_k = k\tau,\ 1 \le k \le N-1,\ N\tau = T, \\ u_0 = \varphi, u_N = \psi, u_l + \left(\frac{\lambda}{\tau} - l\right)(u_{l+1} - u_l) = \xi. \end{cases} \tag{3.2}$$

Here, $l = \left[\frac{\lambda}{\tau}\right]$ and $[\cdot]$ denotes the greatest integer function.

First, we consider the first order of accuracy difference scheme (3.1). As in the differential case for finding a solution $\{u_k\}_{k=1}^{N-1}$ of problem (3.1), we will apply the substitution

$$u_k = v_k + A^{-1}p, \tag{3.3}$$

where $\{v_k\}_{k=0}^{N}$ is the solution of the following auxiliary nonlocal boundary value difference problem. Then, we get

$$\begin{cases} -\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Av_k=\theta_k,\ 1\le k\le N-1, \\ v_0-v_l=\varphi-\xi, v_N-v_l=\psi-\xi, \end{cases} \tag{3.4}$$

where p is the unknown element defined by formulas

$$p=A\varphi-Av_0 \ \text{ or } p=A\psi-Av_N. \tag{3.5}$$

For solving of problem (3.1), we will consider the following algorithm which includes three stages as in algorithm for solving problem (1.1). In the first stage, we consider the auxiliary nonlocal boundary value difference problem (3.4) and we will obtain $\{v_k\}_{k=0}^{N}$.

In the second stage, we will put $k=0$ or $k=N$ and find v_0 or v_N, respectively. Then, using (3.5), we can obtain p. In the third stage, we will use formula (3.3) for obtaining the solution $\{u_k\}_{k=1}^{N-1}$ of difference problem (3.1).

It is well-known (see [28]) that $C=\frac{1}{2}(\tau A+\sqrt{4A+\tau^2A^2})$ is a self-adjoint positive definite operator and $R=(I+\tau C)^{-1}$ which is defined on the whole space H is a bounded operator. Here, I is the identity operator.

Theorem 3.1 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{\theta_k\}_{k=1}^{N-1}\in\mathcal{C}_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $\left(\{u_k\}_{k=1}^{N-1},p\right)$ of difference problem (3.1) the following stability estimates hold:*

$$\left\|\{u_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\le M\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{3.6}$$

$$\left\|A^{-1}p\right\|_H\le M\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{3.7}$$

$$\|p\|_H\le M\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\frac{1}{\alpha(1-\alpha)}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}\right], \tag{3.8}$$

where M does not depend on $\tau,\alpha,\ \varphi,\psi,\xi,$ and $\{\theta_k\}_{k=1}^{N-1}$.
Here, $\mathcal{C}_\tau^{\alpha,\alpha}(H)$ is the space of all H-valued grid functions $\{\theta_k\}_{k=1}^{N-1}$ in the norm

$$\begin{aligned}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)} &= \left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \\ &+\sup_{1\le k<k+n\le N-1}\frac{(k\tau+n\tau)^\alpha(T-k\tau)^\alpha\|\theta_{k+n}-\theta_k\|_H}{(n\tau)^\alpha}.\end{aligned}$$

Proof. We will obtain the representation formula for solution of auxiliary problem (3.4). By using formula (see, [27]), we get

$$\begin{aligned} v_k &= (I-R^{2N})^{-1}[\left(\left(R^k-R^{2N-k}\right)v_0+\left(R^{N-k}-R^{N+k}\right)v_N\right) \\ &-\left(R^{N-k}-R^{N+k}\right)(I+\tau C)(2I+\tau C)^{-1}C^{-1} \\ &\times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau] \\ &+(I+\tau C)(2I+\tau C)^{-1}C^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|}-R^{k+i}\right)\theta_i\tau. \end{aligned} \tag{3.9}$$

Applying formula (3.9) and nonlocal boundary conditions $v_0-v_l=\varphi-\xi$, $v_N-v_l=\psi-\xi$, we get the system of equations

$$\begin{cases} v_0=(I-R^{2N})^{-1}\left(\left(R^l-R^{2N-l}\right)v_0+\left(R^{N-l}-R^{N+l}\right)v_N\right) \\ -(I-R^{2N})^{-1}\left(R^{N-l}-R^{N+l}\right)(I+\tau C)(2I+\tau C)^{-1}C^{-1} \\ \times\sum\limits_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau \\ +(I+\tau C)(2I+\tau C)^{-1}C^{-1}\sum\limits_{i=1}^{N-1}\left(R^{|l-i|}-R^{l+i}\right)\theta_i\tau+\varphi-\xi, \\ v_N=v_0+\psi-\varphi. \end{cases} \tag{3.10}$$

In the similar way as in [16, Lemma 3.2], we can state that the operator

$$Q=I-R^{2N}-R^l+R^{2N-l}-R^{N-l}+R^{N+l}=(I-R^{N-l})(I-R^N)(I-R^l)$$

has inverse and moreover

$$Q^{-1}=\left(I-R^l\right)^{-1}(I-R^N)^{-1}(I-R^{N-l})^{-1}.$$

Solving system (3.10), we have

$$\begin{aligned} v_0 &= -Q^{-1}\left(R^{N-l}-R^{N+l}\right)(I+\tau C)(2I+\tau C)^{-1}C^{-1} \\ &\times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau+Q^{-1}(I-R^{2N})(I+\tau C) \\ &\times(2I+\tau C)^{-1}C^{-1}\sum_{i=1}^{N-1}\left(R^{|l-i|}-R^{l+i}\right)\theta_i\tau \\ &+Q^{-1}(I-R^{2N})(\varphi-\xi)+Q^{-1}\left(R^{N-l}-R^{N+l}\right)(\psi-\varphi), \end{aligned} \tag{3.11}$$

$$v_N = v_0+\psi-\varphi. \tag{3.12}$$

So, difference problem (3.4) has a unique solution $\{v_k\}_{k=0}^N$ defined by formulas (3.9), (3.11), and (3.12). Applying formulas (3.9), (3.11), (3.12) and method of monograph [27], we get

$$\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\le M\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{3.13}$$

$$\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}+\left\|\left\{Av_k\right\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}$$

$$\leq M\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\frac{1}{\alpha(1-\alpha)}\left\|\left\{\theta_k\right\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}\right]. \tag{3.14}$$

The proofs of estimates (3.7), (3.8) are based on formula (3.3), estimates (3.13), (3.14). Applying formula (3.3) and estimates (3.13), (3.7), we can get estimate (3.6). Theorem 3.1 is proved. ■

Theorem 3.2 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{\theta_k\}_{k=1}^{N-1}\in\mathcal{C}_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $\left(\{u_k\}_{k=1}^{N-1},p\right)$ of difference problem (3.1) the following coercive inequality holds:*

$$\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}+\left\|\left\{Au_k\right\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}+\|p\|_H$$

$$\leq M\left[\frac{1}{\alpha(1-\alpha)}\left\|\left\{\theta_k\right\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}+\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H\right], \tag{3.15}$$

where M is independent of $\tau,\alpha,\ \varphi,\psi,\xi,$ and $\{\theta_k\}_{k=1}^{N-1}$.

The proof of estimate (3.15) is based on formula (3.3) and estimates (3.14), (3.8).

Second, we consider the second order of accuracy difference scheme (3.2). We will use the same three stages mentioned above for solving difference problem (3.2).

Applying (3.3) to the second order accuracy difference scheme (3.2) for finding $\{v_k\}_{k=0}^{N}$, we get the following auxiliary difference problem

$$\begin{cases}-\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Av_k=\theta_k,\ 1\leq k\leq N-1,\\ v_0=-\left(\frac{\lambda}{\tau}-l-1\right)v_l+\left(\frac{\lambda}{\tau}-l\right)v_{l+1}+\varphi-\xi,\\ v_N=-\left(\frac{\lambda}{\tau}-l-1\right)v_l+\left(\frac{\lambda}{\tau}-l\right)v_{l+1}+\psi-\xi.\end{cases} \tag{3.16}$$

Theorem 3.3 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{\theta_k\}_{k=1}^{N-1}\in\mathcal{C}_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $\left(\{u_k\}_{k=1}^{N-1},p\right)$ of difference problem (3.2) stability estimates (3.6), (3.7) and (3.8) hold.*

Proof. In the similar way as in [16, Lemma 3.3] we can obtain that, the operator

$$\begin{aligned}Q_1&=I-R^{2N}+\left(\tfrac{\lambda}{\tau}-l-1\right)\left(R^l-R^{2N-l}+R^{N-l}-R^{N+l}\right)\\&-\left(\tfrac{\lambda}{\tau}-l\right)\left(R^{l+1}-R^{2N-l-1}+R^{N-l-1}-R^{N+l+1}\right)\end{aligned}$$

has inverse

$$\begin{aligned}Q_1^{-1}&=(I-R^{2N}+\left(\tfrac{\lambda}{\tau}-l-1\right)\left(R^l-R^{2N-l}+R^{N-l}-R^{N+l}\right)\\&-\left(\tfrac{\lambda}{\tau}-l\right)\left(R^{l+1}-R^{2N-l-1}+R^{N-l-1}-R^{N+l+1}\right))^{-1}.\end{aligned}$$

Applying formula (3.9) for solving auxiliary difference problem (3.16) and nonlocal boundary conditions

$$v_0 = -\left(\tfrac{\lambda}{\tau} - l - 1\right) v_l + \left(\tfrac{\lambda}{\tau} - l\right) v_{l+1} + \varphi - \xi,$$
$$v_N = -\left(\tfrac{\lambda}{\tau} - l - 1\right) v_l + \left(\tfrac{\lambda}{\tau} - l\right) v_{l+1} + \psi - \xi$$

we get the system of equations

$$\begin{cases} v_0 = -(\tfrac{\lambda}{\tau} - l - 1)(I - R^{2N})^{-1}[(R^l - R^{2N-l})v_0 + (R^{N-l} - R^{N+l})v_N] \\ +(\tfrac{\lambda}{\tau} - l - 1)(I - R^{2N})^{-1}(R^{N-l} - R^{N+l})(I + \tau C)(2I + \tau C)^{-1}C^{-1} \\ \times \sum\limits_{i=1}^{N-1} (R^{N-i} - R^{N+i})\theta_i\tau - (\tfrac{\lambda}{\tau} - l - 1)(I + \tau C)(2I + \tau C)^{-1}C^{-1} \\ \times \sum\limits_{i=1}^{N-1} \left(R^{|l-i|} - R^{l+i}\right)\theta_i\tau + \left(\tfrac{\lambda}{\tau} - l\right)(I - R^{2N})^{-1} \\ \times[(R^{l+1} - R^{2N-l-1})v_0 + (R^{N-l-1} - R^{N+l+1})v_N] - (\tfrac{\lambda}{\tau} - l) \\ \times(I - R^{2N})^{-1}(R^{N-l-1} - R^{N+l+1})(I + \tau C)(2I + \tau C)^{-1}C^{-1} \\ \times \sum\limits_{i=1}^{N-1} (R^{N-i} - R^{N+i})\theta_i\tau + (\tfrac{\lambda}{\tau} - l)(I + \tau C)(2I + \tau C)^{-1}C^{-1} \\ \times \sum\limits_{i=1}^{N-1} (R^{|l+1-i|} - R^{l+1+i})\theta_i\tau + \varphi - \xi, \\ v_N = v_0 + \psi - \varphi. \end{cases} \tag{3.17}$$

Solving system (3.17), we get

$$\begin{aligned} v_0 \quad = \quad & (\frac{\lambda}{\tau} - l - 1)Q_1^{-1}(R^{N-l} - R^{N+l})(I + \tau C)(2I + \tau C)^{-1}C^{-1} \\ & \times \sum_{i=1}^{N-1} (R^{N-i} - R^{N+i})\theta_i\tau \\ & -(\frac{\lambda}{\tau} - l - 1)Q_1^{-1}(I - R^{2N})(I + \tau C)(2I + \tau C)^{-1}C^{-1} \\ & \times \sum_{i=1}^{N-1} (R^{|l-i|} - R^{l+i})\theta_i\tau \\ & -(\frac{\lambda}{\tau} - l)Q_1^{-1}(R^{N-l-1} - R^{N+l+1})(I + \tau C)(2I + \tau C)^{-1}C^{-1} \\ & \times \sum_{i=1}^{N-1} (R^{N-i} - R^{N+i})\theta_i\tau \end{aligned}$$

$$
\begin{aligned}
&+(\frac{\lambda}{\tau}-l)Q_1^{-1}(I-R^{2N})(I+\tau C)(2I+\tau C)^{-1}C^{-1} \\
&\times\sum_{i=1}^{N-1}(R^{|l+1-i|}-R^{l+1+i})\theta_i\tau \\
&+Q_1^{-1}(I-R^{2N})(\varphi-\xi)+((\frac{\lambda}{\tau}-l-1)Q_1^{-1}(R^{N-l}-R^{N+l}) \\
&+\left(\frac{\lambda}{\tau}-l\right)Q_1^{-1}\left(R^{N-l-1}-R^{N+l+1}\right))(\psi-\varphi), \qquad (3.18)\\
v_N \;&=\; v_0+\psi-\varphi. \qquad (3.19)
\end{aligned}
$$

So, difference problem (3.16) has a unique solution $\{v_k\}_{k=0}^{N}$ which is defined by formulas (3.9), (3.18), and (3.19). Applying formulas (3.9), (3.18), and (3.19) and method of monograph [27], we get estimates (3.13) and (3.14). The proofs of estimates (3.7), (3.8) for difference problems (3.1) and (3.2) are based on formulas (3.3) and estimates (3.13), (3.14). Applying formula (3.3) and estimates (3.13), (3.7), we can get estimate (3.6). Theorem 3.3 is proved. ■

Theorem 3.4 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{\theta_k\}_{k=1}^{N-1}\in\mathcal{C}_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $\left(\{u_k\}_{k=1}^{N-1},p\right)$ of difference problem (3.2) the coercive inequality (3.15) holds.*

The proof Theorem 3.4 is based on formula (3.16) and estimates (3.14), (3.8).

Now, we give the application of abstract Theorems 3.1-3.4. We consider problem (1.6). We will discretize problem (1.6) into two steps. In the first step, we define the grid spaces

$$
\begin{aligned}
\widetilde{\Omega}_h &= \{x=x_m=(h_1m_1,\cdots,h_nm_n);m=(m_1,\cdots,m_n),\\
m_r &= 0,\cdots,M_r,\ h_rM_r=\ell,r=1,\cdots,n\},
\end{aligned}
$$

$$
\Omega_h=\widetilde{\Omega}_h\cap\Omega, S_h=\widetilde{\Omega}_h\cap S.
$$

To the differential operator A^x generated by problem (1.6), we assign the difference operator A_h^x defined by the formula

$$
A_h^xu^h(x)=-\sum_{r=1}^{n}\left(a_r(x)u_{\bar{x}_r}^h\right)_{x_r,j_r}
$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x)=0$ for all $x\in S_h$. It is well-known that A_h^x is a self-adjoint positive definite operator.

By using A_h^x, for obtaining $v^h(t,x)$ functions, we arrive at auxiliary nonlocal boundary value problem

$$\begin{cases} -\frac{d^2v^h(t,x)}{dt^2}+A_h^x v^h(t,x)=f^h(t,x),\ 0<t<T,\ x\in\Omega_h, \\ v^h(0,x)-v^h(\lambda,x)=\varphi(x)-\xi(x),\ x\in\widetilde{\Omega}_h, \\ v^h(T,x)-v^h(\lambda,x)=\psi(x)-\xi(x), x\in\widetilde{\Omega}_h \end{cases} \tag{3.20}$$

for a system of ordinary differential equations. For calculation of $p^h(x)$, we have formula

$$p^h(x)=A_h^x\varphi^h(x)-A_h^x v^h(0,x), x\in\widetilde{\Omega}_h. \tag{3.21}$$

In the second step, problem (3.20) is replaced by (3.1) and (3.2) as

$$\begin{cases} -\frac{v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)}{\tau^2}+A_h^x v_k^h(x)=f_k^h(x), \\ f_k^h(x)=f^h(t_k,x), t_k=k\tau,\ 1\le k\le N-1,\ x\in\Omega_h, \\ v_0^h(x)-v_l^h(x)=\varphi^h(x)-\xi^h(x),\ x\in\widetilde{\Omega}_h, \\ v_N^h(x)-v_l^h(x)=\psi^h(x)-\xi^h(x), x\in\widetilde{\Omega}_h, \end{cases} \tag{3.22}$$

$$\begin{cases} -\frac{v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)}{\tau^2}+A_h^x v_k^h(x)=f_k^h(x), \\ f_k^h(x)=f^h(t_k,x), t_k=k\tau,\ 1\le k\le N,\ x\in\Omega_h, \\ v_0^h(x)-\left(\frac{\lambda}{\tau}-l\right)v_{l+1}^h(x)+\left(\frac{\lambda}{\tau}-l-1\right)v_l^h(x)=\varphi^h(x)-\xi^h(x), \\ v_N^h(x)-\left(\frac{\lambda}{\tau}-l\right)v_{l+1}^h(x)+\left(\frac{\lambda}{\tau}-l-1\right)v_l^h(x)=\psi^h(x)-\xi^h(x), \\ x\in\widetilde{\Omega}_h. \end{cases} \tag{3.23}$$

To formulate our results, let $L_{2h}=L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2=W_2^2(\widetilde{\Omega}_h)$ be spaces of the grid functions $\rho^h(x)=\{\rho(h_1m_1,\cdots,h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\left\|\rho^h\right\|_{L_{2h}}=(\sum_{x\in\widetilde{\Omega}_h}|\rho^h(x)|^2h_1\cdots h_n)^{1/2},$$

$$\begin{aligned}\left\|\rho^h\right\|_{W_{2h}^2} &= \left\|\rho^h\right\|_{L_{2h}}+(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^n\left|(\rho^h)_{x_r}\right|^2h_1\cdots h_n)^{1/2} \\ &\quad +(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^n\left|(\rho^h(x))_{x_r\overline{x_r},\ m_r}\right|^2h_1\cdots h_n)^{1/2}.\end{aligned}$$

Theorem 3.5 *Let τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ be sufficiently small positive numbers. Then,* *for the solutions of difference schemes (3.1) and (3.2) the following stability estimates hold:*

$$\begin{aligned} \left\| \{u_k^h\}_1^{N-1} \right\|_{\mathcal{C}_\tau(L_{2h})} &\leq M[\|\varphi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}} + \|\xi^h\|_{L_{2h}} + \left\| \{f_k^h\}_1^{N-1} \right\|_{\mathcal{C}_\tau(L_{2h})}], \\ \|p^h\|_{L_{2h}} &\leq M[\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|\xi^h\|_{W_{2h}^2} \\ &+ \frac{1}{\alpha(1-\alpha)} \left\| \{f_k^h\}_1^{N-1} \right\|_{\mathcal{C}_\tau(L_{2h})}], \end{aligned}$$

where M is independent of $\tau, \alpha, h, \varphi^h(x), \psi^h(x), \xi^h(x)$ and $\left\{f_k^h(x)\right\}_1^{N-1}$.

Theorem 3.6 *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers. Then, for the solutions of difference schemes (3.1) and (3.2) the following almost coercive stability estimate holds:*

$$\begin{aligned} &\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^k}{\tau^2}) \right\}_1^{N-1} \right\|_{\mathcal{C}_\tau(L_{2h})} + \left\| \{u_k^h\}_1^{N-1} \right\|_{\mathcal{C}_\tau(W_{2h}^2)} + \|p^h\|_{L_{2h}} \\ \leq\ & M[\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|\xi^h\|_{W_{2h}^2} + \ln \frac{1}{\tau + h} \left\| \{f_k^h\}_1^N \right\|_{\mathcal{C}_\tau(L_{2h})}], \end{aligned}$$

where M does not depend on $\tau, \alpha, h, \varphi^h(x), \psi^h(x), \xi^h(x)$, and $\left\{f_k^h(x)\right\}_1^{N-1}$.

The proofs of Theorems 3.5 and 3.6 are based on the symmetry property of the operator A_h^x in L_{2h} and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 3.7 ([31]) *For the solution of the elliptic difference problem*

$$\begin{cases} A_h^x u^h(x) = \omega^h(x),\ x \in \widetilde{\Omega}_h, \\ u^h(x) = 0,\ x \in S_h, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n} \left\| (u_k^h)_{\overline{x}_r \overline{x}_r, j_r} \right\|_{L_{2h}} \leq M||\omega^h||_{L_{2h}},$$

where M is independent of h and ω.

4 Numerical results

For the numerical result, we consider the inverse problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left((1+x)\frac{\partial u(t,x)}{\partial x}\right) = f(t,x) + p(x), 0 < x < \pi, 0 < t < T, \\ f(t,x) = t\sin(x) - (\exp(-t) + t)(\cos(x) - x\sin(x)), \\ u(0,x) = 2\sin(x), 0 \le x \le \pi, \\ u(T,x) = (\exp(-T) + T + 1)\sin(x), 0 \le x \le \pi, \\ u(\lambda,x) = (\exp(-\lambda) + \lambda + 1)\sin(x), 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le T, \lambda = \frac{3T}{5} \end{cases} \tag{4.1}$$

for the elliptic equation. It is easy to see that $u(t,x) = (\exp(-t) + t + 1)\sin(x)$ and $p(x) = -\cos(x) + (1+x)\sin(x)$ are the exact solutions of (4.1).

For approximate solution of nonlocal boundary value problem (1.3), consider the set $[0,T]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$\begin{aligned} [0,T]_\tau \times [0,\pi]_h \quad = \quad & \{(t_k, x_n) : t_k = k\tau,\ k = 1,\cdots,N-1,\ N\tau = T, \\ & x_n = nh,\ n = 1,\cdots,M-1,\ Mh = \pi\}. \end{aligned}$$

Applying (3.22), we get difference schemes of the first order of accuracy in t and the second order of accuracy in x

$$\begin{cases} \frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} + (1+x_n)\frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + \frac{v_{n+1}^k - v_{n-1}^k}{2h} = \theta_n^k, \\ \theta_n^k = -f(t_k, x_n), k = 1,\cdots,N-1,\ n = 1,\cdots,M-1, \\ v_0^k = v_M^k = 0,\ k = 0,\cdots,N, \\ v_n^0 - v_n^l = \varphi_n - \xi_n,\ n = 0,\cdots,M, \\ v_n^N - v_n^l = \psi_n - \xi_n,\ n = 0,\cdots,M, \\ \varphi_n = \varphi(x_n), \psi_n = \psi(x_n), \xi_n = \xi(x_n), l = \left[\frac{\lambda}{\tau}\right] \end{cases} \tag{4.2}$$

for the approximate solutions of auxiliary nonlocal boundary value problem (1.3) and

$$\begin{cases} \frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + (1+x_n)\frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + \frac{w_{n+1}^k - w_{n-1}^k}{2h} = -p_n, \\ k = 1,\cdots,N-1,\ p_n = p(x_n), n = 1,\cdots,M-1, \\ w_0^k = w_M^k = 0,\ k = 0,\cdots,N, \\ w_n^0 = \xi_n - v_n^l,\ n = 0,\cdots,M, l = \left[\frac{\lambda}{\tau}\right], \\ w_n^N = \xi_n - v_n^l,\ \xi_n = \xi(x_n), n = 0,\cdots,M, \end{cases} \tag{4.3}$$

for the approximate solutions of boundary value problem (1.5).

Applying (3.21) and second order of accuracy in x approximation of A, we get the following values of p in grid points

$$\begin{aligned} p_n &= -(1+x_n)\frac{(\varphi_{n+1}-v_{n+1}^0)-2(\varphi_n-v_n^0)+(\varphi_{n-1}-v_{n-1}^0)}{h^2} \\ &\quad -\frac{(\varphi_{n+1}-v_{n+1}^0)-(\varphi_{n-1}-v_{n-1}^0)}{2h}, n=1,\cdots,M-1. \end{aligned} \tag{4.4}$$

We can rewrite difference scheme (4.2) in the following matrix form

$$\begin{aligned} A_n v_{n+1}+B_n v_n+C_n v_{n-1} &= I\theta_n^k,\ n=1,\cdots,M-1, \\ v_0 &= \overrightarrow{0}, v_M=\overrightarrow{0}. \end{aligned} \tag{4.5}$$

Here, θ_n is $(N+1)\times 1$ column matrix, A_n, B_n, C_n are $(N+1)\times(N+1)$ square matrices

$$A_n=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_n & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_n & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.6}$$

$$B_n=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 & 0 \\ d & b_n & d & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & d & b_n & d & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d & b_n & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & b_n & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & d & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & d & b_n & d \\ 0 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_n & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_n & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.7}$$

where

$$\begin{aligned} a_n &= \frac{1+x_n}{h^2} + \frac{1}{2h},\ b_n = -\frac{2}{\tau^2} - \frac{2(1+x_n)}{h^2}, \\ c_n &= \frac{1+x_n}{h^2} - \frac{1}{2h}, d = \frac{1}{\tau^2}. \end{aligned} \tag{4.8}$$

$$\begin{aligned} \theta_n &= \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix}, \\ \theta_n^0 &= \varphi_n - \xi_n, \theta_n^N = \psi_n - \xi_n,\ n = 1, \cdots, M-1, \\ \theta_n^k &= -f(t_k, x_n),\ k = 1, \cdots, N-1,\ n = 1, \cdots, M-1, \end{aligned}$$

and I is the $(N+1) \times (N+1)$ identity matrix,

$$v_s = \begin{bmatrix} v_s^0 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1}, s = n-1, n, n+1,$$

This type of system is studied by Samarskii and Nikolaev in [32] for difference equations. We seek solution of (4.5) by the formula

$$v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, n = M-1, \cdots, 1,$$

where $v_M = \overrightarrow{0}$, α_n $(n = 1, \cdots, M-1)$ are $(N+1) \times (N+1)$ square matrices and β_n $(n = 1, \cdots, M-1)$ are $(N+1) \times 1$ column matrices. For the solution of difference equation (4.5) we need to use the following formulas for $\alpha_{n+1}, \beta_{n+1}$

$$\begin{aligned} \alpha_{n+1} &= -(B_n + C_n\alpha_n)^{-1}A_n, \\ \beta_{n+1} &= -(B_n + C_n\alpha_n)^{-1}(I\theta_n - C_n\beta_n), n = 1, \cdots, M-1, \end{aligned}$$

where α_1 is the $(N+1)\times(N+1)$ zero matrix and β_1 is the $(N+1)\times 1$ zero column vector.

We can rewrite difference scheme (4.3) in the following matrix form

$$\begin{gathered} A_n w_{n+1} + E_n w_n + C_n w_{n-1} = I\eta_n^k,\ n = 1, \cdots, M-1, \\ w_0 = \overrightarrow{0}, w_M = \overrightarrow{0}. \end{gathered} \tag{4.9}$$

Here, A_n, E_n, C_n are $(N+1)\times(N+1)$ square matrices, A_n and C_n are defined by (4.6) and (4.7),

$$E_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ d & b_n & d & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & d & b_n & d & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d & b_n & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_n & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & d & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & d & b_n & d \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.10}$$

b_n and d are defined by (4.8), η_n is $(N+1)\times 1$ column matrix,

$$\begin{aligned} \eta_n &= \begin{bmatrix} \eta_n^0 \\ \vdots \\ \eta_n^N \end{bmatrix}, \\ \eta_n^0 &= \xi_n - v_n^l, \eta_n^N = \xi_n - v_n^l,\ n = 1, \cdots, M-1, \\ \eta_n^k &= -p_n,\ k = 1, \cdots, N-1,\ n = 1, \cdots, M-1, \end{aligned}$$

$$w_s = \begin{bmatrix} w_s^0 \\ \vdots \\ w_s^N \end{bmatrix}_{(N+1)\times 1}, s = n-1, n, n+1.$$

We seek the solution of (4.9) by the formula

$$w_n = \alpha_{n+1} w_{n+1} + \beta_{n+1}, n = M-1, \cdots, 1,$$

where $w_M = \overrightarrow{0}$, α_n $(n = 1, \cdots, M-1)$ are $(N+1)\times(N+1)$ square matrices and β_n $(n = 1, \cdots, M-1)$ are $(N+1)\times 1$ column matrices. For the solution of difference equation (4.9), we need to use the following formulas for $\alpha_{n+1}, \beta_{n+1}$

$$\begin{aligned}
\alpha_{n+1} &= -(E_n + C_n\alpha_n)^{-1}A_n, \\
\beta_{n+1} &= -(E_n + C_n\alpha_n)^{-1}(I\eta_n - C_n\beta_n), n = 1, \cdots, M-1,
\end{aligned}$$

where α_1 is the $(N+1)\times(N+1)$ zero matrix and β_1 is the $(N+1)\times 1$ zero column vector.

Now, approximate solution of inverse problem will be defined by

$$u_n = v_n + w_n, n = 0, \cdots, M.$$

Second, we again consider inverse problem (4.1). Applying (3.23) and formulas for μ

$$\begin{aligned}
&\frac{\mu(x_{n+1}) - \mu(x_{n-1})}{2h} - \mu'(x_n) = o(h^2), \\
&\frac{\mu(x_{n+1}) - 2\mu(x_n) + \mu(x_{n-1})}{h^2} - \mu''(x_n) = o(h^2), \\
&\frac{2\mu(0) - 5\mu(h) + 4\mu(2h) - \mu(3h)}{h^2} - \mu''(0) = o(h^2), \\
&\frac{2\mu(\pi) - 5\mu(\pi - h) + 4\mu(\pi - 2h) - \mu(\pi - 3h)}{h^2} - \mu''(\pi) = o(h^2),
\end{aligned}$$

we get difference schemes of the second order of accuracy in t and x

$$\begin{cases}
\frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} + (1 + x_n)\frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + \frac{v_{n+1}^k - v_{n-1}^k}{2h} = \theta_n^k, \\
\theta_n^k = -f(t_k, x_n), k = 1, \cdots, N-1,\ n = 1, \cdots, M-1, \\
v_0^k = v_M^k = 0,\ k = 0, \cdots, N, \\
v_n^0 + \left(\frac{\lambda}{\tau} - l - 1\right)v_n^l - \left(\frac{\lambda}{\tau} - l\right)v_n^{l+1} = \varphi_n - \xi_n,\ n = 0, \cdots, M, \\
v_n^N + \left(\frac{\lambda}{\tau} - l - 1\right)v_n^l - \left(\frac{\lambda}{\tau} - l\right)v_n^{l+1} = \psi_n - \xi_n,\ n = 0, \cdots, M, \\
\varphi_n = \varphi(x_n), \psi_n = \psi(x_n), \xi_n = \xi(x_n),
\end{cases} \tag{4.11}$$

for the approximate solutions of boundary value problem (1.3) and

$$\begin{cases}
\frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + (1 + x_n)\frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} - \frac{w_{n+1}^k - w_{n-1}^k}{2h} = -p_n, \\
k = 1, \cdots, N-1,\ p_n = p(x_n), n = 1, \cdots, M-1, \\
w_0^k = w_M^k = 0,\ k = 0, \cdots, N, \\
w_n^0 = \xi_n + \left(\frac{\lambda}{\tau} - l - 1\right)v_n^l - \left(\frac{\lambda}{\tau} - l\right)v_n^{l+1},\ n = 0, \cdots, M, \\
w_n^N = \xi_n + \left(\frac{\lambda}{\tau} - l - 1\right)v_n^l - \left(\frac{\lambda}{\tau} - l\right)v_n^{l+1}, \\
\xi_n = \xi(x_n), n = 0, \cdots, M,
\end{cases} \tag{4.12}$$

for the approximate solutions of nonlocal boundary value problem (1.5).

We can rewrite difference scheme (4.11) in the matrix form (4.5), where A_n, C_n are defined by (4.6), (4.7), (4.8), and B_n will be changed to the following matrix

$$B_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & y & z & 0 & \cdots & 0 & 0 & 0 & 0 \\ d & b_n & d & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & d & b_n & d & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d & b_n & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & b_n & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & d & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & d & b_n & d \\ 0 & 0 & 0 & 0 & \cdots & 0 & y & z & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix},$$

where b_n and d are defined by (4.8),

$$y = \frac{\lambda}{\tau} - l - 1, z = -\frac{\lambda}{\tau} + l.$$

Now we will write difference scheme (4.12) in the matrix form (4.9), where A_n, E_n, C_n are defined by (4.6), (4.10), (4.7), (4.8), and η_n is defined by formula

$$\begin{aligned} \eta_n &= \begin{bmatrix} \eta_n^0 \\ \vdots \\ \eta_n^N \end{bmatrix}, \\ \eta_n^0 &= \xi_n + \left(\frac{\lambda}{\tau} - l - 1\right) v_n^l - \left(\frac{\lambda}{\tau} - l\right) v_n^{l+1}, \\ \eta_n^N &= \xi_n + \left(\frac{\lambda}{\tau} - l - 1\right) v_n^l - \left(\frac{\lambda}{\tau} - l\right) v_n^{l+1},\ n = 1, \cdots, M-1, \\ \eta_n^k &= -p_n,\ k = 1, \cdots, N-1,\ n = 1, \cdots, M-1. \end{aligned}$$

Now we give the results of the numerical analysis. In order to get the approximate solution, we used MATLAB programs. The numerical solutions are recorded for different values of $N = M$ for $T = 2$. Grid functions v_n^k,u_n^k represent the numerical solutions of difference schemes for auxiliary nonlocal problem and inverse problem at (t_k, x_n), respectively. Grid function p_n represents the numerical solutions at x_n for unknown function p. For their comparison, errors are computed by

$$\mathrm{Ev}_M^N = \max_{1\le k\le N-1} \Big(\sum_{n=1}^{M-1} \left|v(t_k,x_n)-v_n^k\right|^2 h\Big)^{\frac{1}{2}},$$
$$\mathrm{Eu}_M^N = \max_{1\le k\le N-1} \Big(\sum_{n=1}^{M-1} \left|u(t_k,x_n)-u_n^k\right|^2 h\Big)^{\frac{1}{2}},$$
$$\mathrm{Ep}_M = \Big(\sum_{n=1}^{M-1} \left|p(x_n)-p_n\right|^2 h\Big)^{\frac{1}{2}}.$$

Tables 1-3 give the error analysis between the exact solutions and solutions derived by difference schemes. Tables 1-3 are constructed for $N = M = 20, 40, 80$ and 160. Hence, the second order of accuracy difference scheme is more accurate comparing to the first order of accuracy difference scheme.

Table 1. Error analysis for nonlocal problem

	N=M=20	N=M=40	N=M=80	N=M=160
Difference scheme (4.2)	0.089063	0.044215	0.022097	0.011055
Difference scheme (4.11)	0.0029957	7.48×10^{-4}	1.87×10^{-4}	4.67×10^{-5}

Table 1 is the error analysis between the exact solution v for auxiliary nonlocal problem and solutions derived by first order and second order accuracy of difference schemes in first stage of algorithm.

Table 2. Error analysis for p

	N=M=20	N=M=40	N=M=80	N=M=160
Difference schemes (4.2,4.4)	0.22429	0.11413	0.057734	0.029054
Difference schemes (4.11,4.4)	0.01924	0.0044386	0.0011099	2.77×10^{-4}

Table 2 is the error analysis between the exact solution p of inverse problem and solutions derived by difference schemes in second stage of algorithm.

Table 3. Error analysis for u

	N=M=20	N=M=40	N=M=80	N=M=160
Difference schemes (4.2,4.4,4.3)	0.053925	0.11413	0.013105	0.0065324
Difference schemes (4.11,4.4,4.12)	3.23×10^{-4}	8.13×10^{-5}	2.03×10^{-5}	5.09×10^{-6}

Table 3 is the error analysis between the exact solution u of inverse problem and solutions derived by first order and second order accuracy of difference schemes.

5 Conclusion

In this work, inverse problem for elliptic equation with Dirichlet condition is considered. The stability and coercive stability estimates for solution of this problem are established. First and second order of accuracy difference schemes are presented for approximate solutions of inverse problem. Theorems on the stability and coercive stability estimates for the solution of difference schemes for elliptic equation are proved. The theoretical statements for the solution of this difference schemes are supported by the results of numerical example in a two dimensional case. As it can be seen from Tables 1-3, second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme. As a future work, higher orders of accuracy difference schemes for the approximate solutions of this problem in an arbitrary Hilbert space E with strongly positive operator A will be investigated.

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