

On approximation of two-dimensional potential and singular operators

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Abstract. The purpose of this paper is the construction of second-order of accuracy quadrature formulas for the numerical calculation of the Vekua types two-dimensional potential and singular integral operators in the unit disk of complex plane. We propose quadrature formulas for these integrals which based on first-order spline approximation of two-dimensional function. MATLAB programs are used for numerical experiments in test examples.

Keywords. Spline approximation, potential integral, singular integral, quadrature formula, numeric integrals.

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1 Introduction

Two-dimensional potential and singular integrals in the circle of complex plane are used to solve widely range of models in applied mechanics and various branches of physics (see [1–13] and references therein).

In paper [3], the first-order of accuracy quadrature formulas for the numerical calculation of the following two-dimensional potential and singular integral operators in the unit disk of complex plane

$$T(\rho|z) = -\frac{1}{\pi} \iint_K \frac{\rho(\zeta)}{\zeta - z} d\zeta, \quad S(\rho|z) = -\frac{1}{\pi} \iint_K \frac{\rho(\zeta)}{(\zeta - z)^2} d\zeta, \quad z \in K,$$

where $K = \{z \in C : |z| \leq 1\}$ are considered. These operators were introduced by I. Vekua [13]. Some properties of operators $T(\rho|z)$ and $S(\rho|z)$ are described in [10, p. 198-210]. In particular,

$$\frac{\partial T(\rho|z)}{\partial z} = S(\rho|z), \quad \frac{\partial T(\rho|z)}{\partial \bar{z}} = \rho(z), \quad z \in K \quad (1)$$

are given. Our purpose in this paper is the construction of second-order of accuracy quadrature formulas for the numerical calculation of the operators $T(\rho|z)$ and $S(\rho|z)$.

Following [3], we will use grid sets for the unit disk K in the complex plane. First, we introduce the following notations:

$$\begin{aligned} [0, 1]_\tau &= \{r_k \mid r_k = k\tau, 1 \leq k \leq N, N\tau = 1\}, \\ [-\pi, \pi]_{h_k} &= \{\theta_{k,m} \mid \theta_{k,m} = -\pi + mh_k, 0 \leq m \leq M_k, \\ M_k h_k &= 2\pi, M_k = 2k + 1, 1 \leq k \leq N\}; K_\tau = \{z \in C : |z| < \tau\}, \\ K_{r_k} &= \{z \in C : |z| < r_k\}, 1 \leq k \leq N; H_{r_k} = K_{r_{k+1}} - K_{r_k}. \end{aligned}$$

These notations permit us to write

$$K = K_\tau \bigcup_{k=1}^{N-1} \bigcup_{m=0}^{M_k-1} H_{r_k} = K_\tau \bigcup_{k=1}^{N-1} \bigcup_{m=0}^{M_k-1} D_{k,m},$$

where domain $D_{k,m}$ is defined by

$$\begin{aligned} D_{k,m} &= \{\zeta \mid \zeta = r e^{i\theta}, r_k \leq r \leq r_{k+1}, 1 \leq k \leq N-1, \\ &\theta_{k,m} \leq \theta \leq \theta_{k,m+1}, 0 \leq m \leq M_k - 1\} \end{aligned}$$

with boundary $\partial D_{k,m}$:

$$\begin{aligned} \partial D_{k,m} &= \Gamma_{km} = \Gamma_{km}^1 \cup \Gamma_{km}^2 \cup \Gamma_{km}^3 \cup \Gamma_{km}^4, \\ \Gamma_{km}^1 &= \{\zeta \mid \zeta = r e^{i\theta_{k,m+1}}, r_k \leq r \leq r_{k+1}\}, \\ \Gamma_{km}^2 &= \{\zeta \mid \zeta = r_{k+1} e^{i\theta}, \theta_{k,m+1} \geq \theta \geq \theta_{k,m}\}, \\ \Gamma_{km}^3 &= \{\zeta \mid \zeta = r e^{i\theta_{k,m}}, r_{k+1} \geq r \geq r_k\}, \\ \Gamma_{km}^4 &= \{\zeta \mid \zeta = r_k e^{i\theta}, \theta_{k,m} \leq \theta \leq \theta_{k,m+1}\} \end{aligned}$$

and its four corner points are

$$\begin{aligned} z_{0,0}^{k,m} &= r_k e^{i\theta_{k,m}}, z_{1,0}^{k,m} = r_{k+1} e^{i\theta_{k,m}}, \\ z_{1,1}^{k,m} &= r_{k+1} e^{i\theta_{k,m+1}}, z_{0,1}^{k,m} = r_k e^{i\theta_{k,m+1}} \end{aligned} \tag{2}$$

for any $0 \leq m \leq M_k, 1 \leq k \leq N$.

Second, we introduce grid points

$$\begin{aligned} z_{0,0}^{*k,m} &= (r_k + \frac{\tau}{2}) e^{i(\theta_{k,m} + \frac{h_k}{2})}, z_{1,0}^{*k,m} = (r_{k+1} + \frac{\tau}{2}) e^{i(\theta_{k,m} + \frac{h_k}{2})}, \\ z_{1,1}^{*k,m} &= (r_{k+1} + \frac{\tau}{2}) e^{i(\theta_{k,m+1} + \frac{h_k}{2})}, z_{0,1}^{*k,m} = (r_k + \frac{\tau}{2}) e^{i(\theta_{k,m+1} + \frac{h_k}{2})} \end{aligned}$$

in which the approximate values of T and S will be calculated and denote by $D_{k,m}^*$ corresponding small domains with four corner points. Thus, we have two sets of gridpoints $\Omega^{(1)}$ and $\Omega^{(2)}$ defined by

$$\begin{aligned}\Omega^{(1)} &= \Omega_{\tau,h}^{(1)} \\ &= \left\{ z_{0,0}^{k,m}, z_{1,0}^{k,m}, z_{1,1}^{k,m}, z_{0,1}^{k,m} \mid 1 \leq k \leq N-1, 0 \leq m \leq M_k-1 \right\}, \\ \Omega^{(2)} &= \Omega_{\tau,h}^{(2)} \\ &= \left\{ z_{0,0}^{*k,m}, z_{1,0}^{*k,m}, z_{1,1}^{*k,m}, z_{0,1}^{*k,m} \mid 1 \leq k \leq N-1, 0 \leq m \leq M_k-1 \right\}.\end{aligned}$$

2 Spline approximation

In this section, we describe approximation of complex valued function ρ by the first-order splines in two variables z and \bar{z} . From $z = x + iy$, $\bar{z} = x - iy$, it follows $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$. So, applying first-order two dimensional splines in variables x and y , we have the following approximation (see [2, p.12, formula (1.16)]):

$$\rho(z) \approx \hat{\rho}(z) = a_{km}z + b_{km}\bar{z} + c_{km}(z^2 - \bar{z}^2) + d_{km}, \quad z \in D_{k,m}, \quad (3)$$

where unknown coefficients $a_{km}, b_{km}, c_{km}, d_{km}$ are defined by given values of ρ at four corner points (2). Hence, we get linear system equations

$$\begin{aligned}\rho\left(z_{0,0}^{k,m}\right) &= a_{km} z_{0,0}^{k,m} + b_{km} \overline{z_{0,0}^{k,m}} + c_{km} \left[\left(z_{0,0}^{k,m}\right)^2 - \overline{z_{0,0}^{k,m}}^2 \right] + d_{km}, \\ \rho\left(z_{1,0}^{k,m}\right) &= a_{km} z_{1,0}^{k,m} + b_{km} \overline{z_{1,0}^{k,m}} + c_{km} \left[\left(z_{1,0}^{k,m}\right)^2 - \overline{z_{1,0}^{k,m}}^2 \right] + d_{km}, \\ \rho\left(z_{1,1}^{k,m}\right) &= a_{km} z_{1,1}^{k,m} + b_{km} \overline{z_{1,1}^{k,m}} + c_{km} \left[\left(z_{1,1}^{k,m}\right)^2 - \overline{z_{1,1}^{k,m}}^2 \right] + d_{km}, \\ \rho\left(z_{0,1}^{k,m}\right) &= a_{km} z_{0,1}^{k,m} + b_{km} \overline{z_{0,1}^{k,m}} + c_{km} \left[\left(z_{0,1}^{k,m}\right)^2 - \overline{z_{0,1}^{k,m}}^2 \right] + d_{km}.\end{aligned} \quad (4)$$

Solving (2), we obtain coefficients a_{km}, b_{km}, c_{km} , and d_{km} as follows:

$$\begin{aligned}a_{k,m} &= Q_{k,m}^{-1} \cdot \left(r_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} - r_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \right. \\ &\quad \left. - r_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} + r_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} + r_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \right.\end{aligned}$$

$$\begin{aligned}
 & -r_{1,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} - r_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + r_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
 & + r_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} - r_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
 & + r_{1,1}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} + r_{1,1}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} - r_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & - r_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + r_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} + r_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & - r_{1,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + r_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & + r_{1,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} - r_{1,1}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & + r_{1,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} + r_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 - r_{0,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & - r_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 + r_{0,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} + r_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \\
 & - r_{1,0}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 - r_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{0,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & + r_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{0,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \\
 & + r_{1,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{0,1}^{k,m}} + r_{0,0}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{0,0}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & - r_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{1,0}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + r_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \\
 & - r_{1,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} - r_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{0,1}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & + r_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{1,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \\
 & + r_{1,1}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \Big), \tag{5}
 \end{aligned}$$

$$\begin{aligned}
 b_{k,m} &= Q_{k,m}^{-1} \cdot \left(r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 - r_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \right. \\
 & - r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 + r_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} + r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \\
 & - r_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{0,1}^{k,m} - r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 + r_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \\
 & \left. + r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 - r_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} - r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \right)
 \end{aligned}$$

$$\begin{aligned}
& +r_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{0,1}^{k,m} + r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 - r_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \\
& -r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 + r_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} + r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \\
& -r_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} - r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 + r_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \\
& +r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 - r_{1,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} - r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \\
& +r_{1,1}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} - r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 + r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \\
& +r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 - r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 - r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \\
& +r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 + r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{0,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \\
& -r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{0,1}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 + r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \\
& -r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 - r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{0,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \\
& +r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{1,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 - r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \\
& +r_{1,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 + r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{0,1}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \\
& -r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{1,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 + r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \\
& -r_{1,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2), \tag{6}
\end{aligned}$$

$$\begin{aligned}
c_{k,m} = & -Q_{k,m}^{-1} \cdot \left(r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} - r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} - r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \right. \\
& +r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} + r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} - r_{0,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} - r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,1}^{k,m}} \\
& +r_{0,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} + r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} - r_{0,1}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} - r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \\
& +r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} + r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} - r_{0,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} - r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \\
& +r_{1,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} + r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{1,0}^{k,m}} - r_{1,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} - r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \\
& +r_{0,1}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} + r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,1}^{k,m}} - r_{1,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} - r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \\
& \left. +r_{1,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}}\right), \tag{7}
\end{aligned}$$

$$\begin{aligned}
 d_{k,m} = & Q_{k,m}^{-1} \cdot \left(r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m} \right)^2 \cdot \overline{z_{1,0}^{k,m}} \right. \\
 & + r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m} \right)^2 \cdot \overline{z_{0,1}^{k,m}} - r_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} \\
 & + r_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} - r_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m} \right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \\
 & - r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} + r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m} \right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & - r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m} \right)^2 \cdot \overline{z_{0,0}^{k,m}} + r_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} \\
 & - r_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} + r_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m} \right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
 & + r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m} \right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
 & + r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m} \right)^2 \cdot \overline{z_{0,0}^{k,m}} - r_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,1}^{k,m}} \\
 & + r_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} - r_{1,0}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
 & - r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot \overline{z_{1,0}^{k,m}} + r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m} \right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
 & - r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m} \right)^2 \cdot \overline{z_{0,0}^{k,m}} + r_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \\
 & - r_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} + r_{1,1}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
 & + r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \cdot \left(z_{1,1}^{k,m} \right)^2 - r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & - r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(z_{1,1}^{k,m} \right)^2 + r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & + r_{0,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(z_{1,0}^{k,m} \right)^2 - r_{0,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & - r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \cdot \left(z_{1,1}^{k,m} \right)^2 + r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & + r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(z_{1,1}^{k,m} \right)^2 - r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & - r_{0,1}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(z_{1,0}^{k,m} \right)^2 + r_{0,1}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & + r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(z_{1,1}^{k,m} \right)^2 - r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & - r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(z_{1,1}^{k,m} \right)^2 + r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
 & + r_{1,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(z_{0,1}^{k,m} \right)^2 - r_{1,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
 & - r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(z_{1,0}^{k,m} \right)^2 + r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m} \right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & + r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(z_{0,1}^{k,m} \right)^2 - r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
 & \left. - r_{1,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(z_{0,1}^{k,m} \right)^2 + r_{1,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m} \right)^2 \cdot \overline{z_{0,1}^{k,m}} \right), \tag{8}
 \end{aligned}$$

where

$$\begin{aligned}
 Q_{k,m} = & z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} - z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} + z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} \\
 & - \left(z_{0,0}^{k,m}\right)^2 \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} + \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} - \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
 & - z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} - z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} \\
 & + \left(z_{0,0}^{k,m}\right)^2 \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,1}^{k,m}} - \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} + \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
 & + z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} + z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} \\
 & - \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} + \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} - \left(z_{1,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
 & - z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} - z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
 & + \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} - \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \\
 & + \left(z_{1,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} + z_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{\left(z_{1,0}^{k,m}\right)^2} - z_{0,0}^{k,m} \cdot \overline{\left(z_{0,1}^{k,m}\right)^2} \cdot \overline{z_{1,0}^{k,m}} \\
 & - z_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{\left(z_{1,0}^{k,m}\right)^2} + z_{0,1}^{k,m} \cdot \overline{\left(z_{0,0}^{k,m}\right)^2} \cdot \overline{z_{1,0}^{k,m}} + z_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{\left(z_{0,1}^{k,m}\right)^2} \\
 & - z_{1,0}^{k,m} \cdot \overline{\left(z_{0,0}^{k,m}\right)^2} \cdot \overline{z_{0,1}^{k,m}} - z_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{\left(z_{1,1}^{k,m}\right)^2} + z_{0,0}^{k,m} \cdot \overline{\left(z_{0,1}^{k,m}\right)^2} \cdot \overline{z_{1,1}^{k,m}} \\
 & + z_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{\left(z_{1,1}^{k,m}\right)^2} - z_{0,1}^{k,m} \cdot \overline{\left(z_{0,0}^{k,m}\right)^2} \cdot \overline{z_{1,1}^{k,m}} - z_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{\left(z_{0,1}^{k,m}\right)^2} \\
 & z_{1,1}^{k,m} \cdot \overline{\left(z_{0,0}^{k,m}\right)^2} \cdot \overline{z_{0,1}^{k,m}} + z_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{\left(z_{1,1}^{k,m}\right)^2} - z_{0,0}^{k,m} \cdot \overline{\left(z_{1,0}^{k,m}\right)^2} \cdot \overline{z_{1,1}^{k,m}} \\
 & - z_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{\left(z_{1,1}^{k,m}\right)^2} + z_{1,0}^{k,m} \cdot \overline{\left(z_{0,0}^{k,m}\right)^2} \cdot \overline{z_{1,1}^{k,m}} + z_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{\left(z_{1,0}^{k,m}\right)^2} \\
 & - z_{1,1}^{k,m} \cdot \overline{\left(z_{0,0}^{k,m}\right)^2} \cdot \overline{z_{1,0}^{k,m}} - z_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{\left(z_{1,1}^{k,m}\right)^2} + z_{0,1}^{k,m} \cdot \overline{\left(z_{1,0}^{k,m}\right)^2} \cdot \overline{z_{1,1}^{k,m}} \\
 & + z_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{\left(z_{1,1}^{k,m}\right)^2} - z_{1,0}^{k,m} \cdot \overline{\left(z_{0,1}^{k,m}\right)^2} \cdot \overline{z_{1,1}^{k,m}} - z_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{\left(z_{1,0}^{k,m}\right)^2} \\
 & + z_{1,1}^{k,m} \cdot \overline{\left(z_{0,1}^{k,m}\right)^2} \cdot \overline{z_{1,0}^{k,m}} \Big).
 \end{aligned}$$

3 Approximation of T

In this section, we give quadrature formula for potential integral T . By using spline approximation of ρ in the domain $D_{k,m}$, we have

$$\begin{aligned}
 T(\widehat{\rho}|z) = & -\frac{1}{\pi} \sum_{k=1}^{N-1} \sum_{m=0}^{M_k-1} \left(a_{k,m} \iint_{D_{k,m}} \frac{\zeta d\zeta}{\zeta-z} + b_{k,m} \iint_{D_{k,m}} \frac{\overline{\zeta} d\overline{\zeta}}{\overline{\zeta}-z} \right. \\
 & \left. + c_{k,m} \iint_{D_{k,m}} \frac{(\zeta^2 - \overline{\zeta}^2) d\zeta}{\zeta-z} + d_{k,m} \iint_{D_{k,m}} \frac{d\zeta}{\zeta-z} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^{N-1} \sum_{m=0}^{M_k-1} \left(a_{k,m} T_{k,m}^a(z) + b_{k,m} T_{k,m}^b(z) \right. \\
 &\quad \left. + c_{k,m} T_{k,m}^c(z) + d_{k,m} T_{k,m}^d(z) \right), \tag{9}
 \end{aligned}$$

where

$$\begin{aligned}
 T_{k,m}^a(z) &= -\frac{1}{\pi} \iint_{D_{k,m}} \frac{\zeta d\zeta}{\zeta-z}, \quad T_{k,m}^b(z) = -\frac{1}{\pi} \iint_{D_{k,m}} \frac{\bar{\zeta} d\zeta}{\zeta-z}, \\
 T_{k,m}^c(z) &= -\frac{1}{\pi} \iint_{D_{k,m}} \frac{(\zeta^2-\bar{\zeta}^2) d\zeta}{\zeta-z}, \quad T_{k,m}^d(z) = -\frac{1}{\pi} \iint_{D_{k,m}} \frac{d\zeta}{\zeta-z}.
 \end{aligned}$$

Let us evaluate $T_{k,m}^a$. By using Pompeu formula (see [13, p. 26, formula (4.15)]), two-dimensional integral can be reduced to one dimensional integral on boundary $\Gamma_{k,m}$. From identity $\frac{\partial(\zeta\bar{\zeta})}{\partial\zeta} = \bar{\zeta}$ on $\Gamma_{k,m}^1$, it follows that

$$T_{k,m}^a(z) = \int_{\Gamma_{k,m}} \frac{\zeta\bar{\zeta}}{\zeta-z} d\zeta = \sum_{j=1}^4 \int_{\Gamma_{k,m}^j} \frac{\zeta\bar{\zeta}}{\zeta-z} d\zeta. \tag{10}$$

In sequel,

$$\begin{aligned}
 T_{k,m}^{a,1}(z) &= \int_{\Gamma_{k,m}^1} \frac{\zeta\bar{\zeta}}{\zeta-z} d\zeta = e^{-2i\theta_{k,m}} \int_{\Gamma_{k,m}^1} \frac{\zeta^2}{\zeta-z} d\zeta \\
 &= e^{-2i\theta_{k,m}} \int_{\Gamma_{k,m}^1} \frac{\zeta^2-z^2+z^2}{\zeta-z} d\zeta \\
 &= e^{-2i\theta_{k,m}} \left[\int_{\Gamma_{k,m}^1} \frac{\zeta^2-z^2}{\zeta-z} d\zeta + z^2 \int_{\Gamma_{k,m}^1} \frac{d\zeta}{\zeta-z} \right] \\
 &= e^{-2i\theta_{k,m}} \left[\frac{\zeta^2}{2} + z\zeta + z^2 \ln(\zeta-z) \right]_{z_{0,0}^{k,m}}^{z_{1,0}^{k,m}} \\
 &= \frac{\bar{z}_{0,0}^{k,m}}{z_{0,0}^{k,m}} \left[\frac{(z_{1,0}^{k,m})^2}{2} - \frac{(z_{0,0}^{k,m})^2}{2} + z z_{1,0}^{k,m} - z z_{0,0}^{k,m} + z^2 \ln \left| \frac{z_{1,0}^{k,m}-z}{z_{0,0}^{k,m}-z} \right| \right].
 \end{aligned} \tag{11}$$

We have $\zeta\bar{\zeta} = r_{k+1} e^{i\theta} r_{k+1} e^{-i\theta} = r_{k+1}^2$ on $\Gamma_{k,m}^2$. Thus,

$$\begin{aligned}
 T_{k,m}^{a,2}(z) &= \int_{\Gamma_{k,m}^2} \frac{\zeta\bar{\zeta}}{\zeta-z} d\zeta = r_{k+1}^2 \int_{\Gamma_{k,m}^2} \frac{d\zeta}{\zeta-z} \\
 &= r_{k+1}^2 \ln(\zeta-z) \Big|_{z_{1,0}^{k,m}}^{z_{1,1}^{k,m}} = z_{1,0}^{k,m} \overline{z_{1,0}^{k,m}} \ln \frac{z_{1,1}^{k,m}-z}{z_{1,0}^{k,m}-z}.
 \end{aligned} \tag{12}$$

For any $\zeta \in \Gamma_{k,m}^3$, the relation $\zeta\bar{\zeta} = r e^{i\theta_{k,m}} r e^{-i\theta_{k,m}} = r^2 = \zeta^2 e^{-2i\theta_{k,m}}$ is true. So, we get

$$\begin{aligned} T_{k,m}^{a,3}(z) &= \int_{\Gamma_{k,m}^3} \frac{\zeta\bar{\zeta}}{\zeta-z} d\zeta = e^{-2i\theta_{k,m}} \int_{\Gamma_{k,m}^3} \frac{\zeta^2 - z^2 + z^2}{\zeta-z} d\zeta \\ &= e^{-2i\theta_{k,m}} \left(\int_{\Gamma_{k,m}^3} (\zeta+z) d\zeta + \int_{\Gamma_{k,m}^3} \frac{z^2}{\zeta-z} d\zeta \right) \\ &= \frac{\overline{z_{1,1}^{k,m}}}{z_{1,1}^{k,m}} \left[\frac{(z_{0,1}^{k,m})^2}{2} - \frac{(z_{1,1}^{k,m})^2}{2} + z z_{0,1}^{k,m} - z z_{1,1}^{k,m} + z^2 \ln \left| \frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} \right| \right]. \end{aligned} \quad (13)$$

Since $\zeta\bar{\zeta} = r_k e^{i\theta} r_k e^{-i\theta} = r_k^2 = z_{k,m} \overline{z_{k,m}}$ on $\Gamma_{k,m}^4$, we get

$$\begin{aligned} T_{k,m}^{a,4}(z) &= \int_{\Gamma_{k,m}^4} \frac{\zeta\bar{\zeta}}{\zeta-z} d\zeta = r_k^2 \int_{\Gamma_{k,m}^4} \frac{d\zeta}{\zeta-z} \\ &= r_k^2 (\ln(\zeta-z))_{z_{0,1}^{k,m}}^{z_{0,0}^{k,m}} = z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}} \ln \frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z}. \end{aligned} \quad (14)$$

By virtue (10)- (14), we have formula

$$\begin{aligned} T_{k,m}^a(z) &= \frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}} \left[\frac{(z_{1,0}^{k,m})^2}{2} - \frac{(z_{0,0}^{k,m})^2}{2} + z z_{1,0}^{k,m} - z z_{0,0}^{k,m} + z^2 \ln \left| \frac{z_{1,0}^{k,m} - z}{z_{0,0}^{k,m} - z} \right| \right] \\ &\quad + z_{1,0}^{k,m} \overline{z_{1,0}^{k,m}} \ln \frac{z_{1,1}^{k,m} - z}{z_{1,0}^{k,m} - z} + z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}} \ln \frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z} \\ &\quad + \frac{\overline{z_{1,1}^{k,m}}}{z_{1,1}^{k,m}} \left[\frac{z_{0,1}^{k,m}}{2} - \frac{(z_{1,1}^{k,m})^2}{2} + z z_{0,1}^{k,m} - z z_{1,1}^{k,m} + z^2 \ln \left| \frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} \right| \right]. \end{aligned} \quad (15)$$

In a similar manner, the following formulas for

$T_{k,m}^b(z)$, $T_{k,m}^c(z)$, $T_{k,m}^d(z)$ can be obtained respectively:

$$\begin{aligned} T_{k,m}^b(z) &= \frac{1}{2} \left(\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}} \right)^2 \\ &\quad \times \left[\frac{(z_{1,0}^{k,m})^2}{2} - \frac{(z_{0,0}^{k,m})^2}{2} + z z_{1,0}^{k,m} - z z_{0,0}^{k,m} + z^2 \ln \left| \frac{z_{1,0}^{k,m} - z}{z_{0,0}^{k,m} - z} \right| \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \left(\overline{z_{1,1}^{k,m}} z_{1,1}^{k,m} \right)^2 \left[\frac{1}{z^2} \ln \left(\frac{z_{1,1}^{k,m} - z}{z_{1,0}^{k,m} - z} z_{1,0}^{k,m} + \frac{1}{z} \left(\frac{1}{z_{1,1}^{k,m}} - \frac{1}{z_{1,0}^{k,m}} \right) \right) \right] \\
 & + \frac{1}{2} \left(\overline{z_{0,1}^{k,m}} z_{0,1}^{k,m} \right)^2 \left[\frac{(z_{0,1}^{k,m})^2}{2} - \frac{(z_{1,1}^{k,m})^2}{2} + z z_{0,1}^{k,m} - z z_{1,1}^{k,m} + z^2 \ln \frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} \right] \\
 & + \frac{1}{2} \left(\overline{z_{0,0}^{k,m}} z_{0,0}^{k,m} \right)^2 \left[\frac{1}{z^2} \ln \left(\frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z} z_{0,1}^{k,m} + \frac{1}{z} \left(\frac{1}{z_{0,0}^{k,m}} - \frac{1}{z_{0,1}^{k,m}} \right) \right) \right],
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 T_{k,m}^c(z) & = \left(\left(\overline{z_{0,0}^{k,m}} z_{0,0}^{k,m} \right) - \frac{1}{3} \left(\overline{z_{0,0}^{k,m}} z_{0,0}^{k,m} \right)^3 \right) \left[\frac{(z_{1,0}^{k,m})^3}{3} - \frac{(z_{0,0}^{k,m})^3}{3} + z \frac{(z_{1,0}^{k,m})^2}{2} \right. \\
 & \quad \left. - z \frac{(z_{0,0}^{k,m})^2}{2} + z^2 (z_{1,0}^{k,m} - z_{0,0}^{k,m}) + z^3 \ln \frac{z_{1,0}^{k,m} - z}{z_{0,0}^{k,m} - z} \right] \\
 & + (z_{1,0}^{k,m} \overline{z_{1,0}^{k,m}}) \left\{ z_{1,1}^{k,m} - z_{1,0}^{k,m} + z \ln \left(\frac{z_{1,1}^{k,m} - z}{z_{1,0}^{k,m} - z} \right) \right\} - \frac{(z_{1,0}^{k,m} \overline{z_{1,0}^{k,m}})^3}{3} \\
 & \times \left\{ -\frac{1}{z^3} \cdot \ln \left(\frac{z_{1,1}^{k,m}}{z_{1,0}^{k,m}} \right) + \frac{1}{z^2} \cdot \left(\frac{1}{z_{1,1}^{k,m}} - \frac{1}{z_{1,0}^{k,m}} \right) \right. \\
 & \quad \left. + \frac{1}{2z} \cdot \left(\frac{1}{(z_{1,1}^{k,m})^2} - \frac{1}{(z_{1,0}^{k,m})^2} \right) + \frac{1}{z^3} \cdot \ln \left(\frac{z_{1,1}^{k,m} - z}{z_{1,0}^{k,m} - z} \right) \right\} \\
 & + \left(\left(\overline{z_{0,1}^{k,m}} z_{0,1}^{k,m} \right) - \frac{1}{3} \left(\overline{z_{0,1}^{k,m}} z_{0,1}^{k,m} \right)^3 \right) \left[\frac{(z_{0,1}^{k,m})^3}{3} - \frac{(z_{1,1}^{k,m})^3}{3} + z \frac{(z_{0,1}^{k,m})^2}{2} \right. \\
 & \quad \left. - z \frac{(z_{1,1}^{k,m})^2}{2} + z^2 (z_{0,1}^{k,m} - z_{1,1}^{k,m}) + z^3 \ln \left| \frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} \right| \right] \\
 & + (z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}}) \left\{ z_{0,0}^{k,m} - z_{0,1}^{k,m} + z \ln \left(\frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z} \right) \right\} - \frac{(z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}})^3}{3} \\
 & \times \left\{ -\frac{1}{z^3} \cdot \ln \left(\frac{z_{0,0}^{k,m}}{z_{0,1}^{k,m}} \right) + \frac{1}{z^2} \cdot \left(\frac{1}{z_{0,0}^{k,m}} - \frac{1}{z_{0,1}^{k,m}} \right) \right. \\
 & \quad \left. + \frac{1}{2z} \cdot \left(\frac{1}{(z_{0,0}^{k,m})^2} - \frac{1}{(z_{0,1}^{k,m})^2} \right) + \frac{1}{z^3} \cdot \ln \left(\frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z} \right) \right\},
 \end{aligned} \tag{17}$$

$$\begin{aligned}
 T_{k,m}^d(z) & = \frac{\overline{z_{0,0}^{k,m}} z_{0,0}^{k,m}}{z_{0,0}^{k,m}} \left[z_{1,0}^{k,m} - z_{0,0}^{k,m} + z \ln \frac{z_{1,0}^{k,m} - z}{z_{0,0}^{k,m} - z} \right] \\
 & + z_{1,1}^{k,m} \overline{z_{1,1}^{k,m}} \frac{1}{z} \ln \frac{(z_{1,1}^{k,m} - z) z_{1,0}^{k,m}}{(z_{1,0}^{k,m} - z) z_{1,1}^{k,m}} \\
 & + \frac{\overline{z_{1,1}^{k,m}} z_{1,1}^{k,m}}{z_{1,1}^{k,m}} \left[z_{0,1}^{k,m} - z_{1,1}^{k,m} + z \ln \frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} \right] + \frac{z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}}}{z} \ln \left(\frac{(z_{0,0}^{k,m} - z) z_{0,1}^{k,m}}{(z_{0,1}^{k,m} - z) z_{0,0}^{k,m}} \right).
 \end{aligned} \tag{18}$$

Finally, we get an approximate formula

$$T(\rho|z) \approx \sum_{k=1}^{N-1} \sum_{m=0}^{M_k-1} \left(a_{k,m} T_{k,m}^a(z) + b_{k,m} T_{k,m}^b(z) \right. \\ \left. + c_{k,m} T_{k,m}^c(z) + d_{k,m} T_{k,m}^d(z) \right), z \in \Omega^{(2)},$$

where coefficients $a_{k,m}$, $b_{k,m}$, $c_{k,m}$, $d_{k,m}$ and integrals $T_{k,m}^a$, $T_{k,m}^b$, $T_{k,m}^c$, and $T_{k,m}^d$ are defined by (5), (6), (7), (8) and (15), (16), (17), (18), respectively.

4 Approximation of S

In this section, we present quadrature formula for singular integral S . Applying spline approximation of ρ in the domain $D_{k,m}$, we have

$$S(\widehat{\rho}|z) = -\frac{1}{\pi} \sum_{k=1}^{N-1} \sum_{m=0}^{M_k-1} \left(a_{k,m} S_{k,m}^a(z) + b_{k,m} S_{k,m}^b(z) \right. \\ \left. + c_{k,m} S_{k,m}^c(z) + d_{k,m} S_{k,m}^d(z) \right),$$

where

$$S_{k,m}^a(z) = -\frac{1}{\pi} \iint_{D_{k,m}} \frac{\zeta d\zeta}{(\zeta-z)^2}, \quad S_{k,m}^b(z) = -\frac{1}{\pi} \iint_{D_{k,m}} \frac{\bar{\zeta} d\zeta}{(\zeta-z)^2}, \\ S_{k,m}^c(z) = -\frac{1}{\pi} \iint_{D_{k,m}} \frac{(\zeta^2 - \bar{\zeta}^2) d\zeta}{(\zeta-z)^2}, \quad S_{k,m}^d(z) = -\frac{1}{\pi} \iint_{D_{k,m}} \frac{d\zeta}{(\zeta-z)^2}.$$

From (1), it follows that

$$S_{k,m}^a(z) = \frac{\partial T_{k,m}^a(z)}{\partial z}, \quad S_{k,m}^b(z) = \frac{\partial T_{k,m}^b(z)}{\partial z}, \\ S_{k,m}^c(z) = \frac{\partial T_{k,m}^c(z)}{\partial z}, \quad S_{k,m}^d(z) = \frac{\partial T_{k,m}^d(z)}{\partial z}.$$

Applying (15), (16), (17) and (18), we get

$$S_{k,m}^a(z) = \frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}} \\ \times \left[z_{1,0}^{k,m} - z_{0,0}^{k,m} + 2z \ln \left(\frac{z_{1,0}^{k,m} - z}{z_{0,0}^{k,m} - z} \right) - z^2 \left(\frac{1}{z_{1,0}^{k,m} - z} - \frac{1}{z_{0,0}^{k,m} - z} \right) \right] \\ + z_{1,0}^{k,m} \overline{z_{1,0}^{k,m}} \left[\frac{1}{z_{1,0}^{k,m} - z} - \frac{1}{z_{1,1}^{k,m} - z} \right] + z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}} \left[\frac{1}{z_{0,1}^{k,m} - z} - \frac{1}{z_{0,0}^{k,m} - z} \right] \\ + \frac{\overline{z_{1,1}^{k,m}}}{z_{1,1}^{k,m}} \left[z_{0,1}^{k,m} - z_{1,1}^{k,m} + 2z \ln \left(\frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} \right) - z^2 \left(\frac{1}{z_{0,1}^{k,m} - z} - \frac{1}{z_{1,1}^{k,m} - z} \right) \right], \quad (19)$$

$$\begin{aligned}
 S_{k,m}^b(z) &= \frac{1}{2} \left(\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}} \right)^2 \\
 &\times \left[z_{1,0}^{k,m} - z_{0,0}^{k,m} + 2z \ln \left(\frac{z_{1,0}^{k,m} - z}{z_{0,0}^{k,m} - z} \right) + z^2 \left(\frac{1}{z_{0,0}^{k,m} - z} - \frac{1}{z_{1,0}^{k,m} - z} \right) \right] \\
 &+ \frac{1}{2} \left(\frac{\overline{z_{1,1}^{k,m}}}{z_{1,1}^{k,m}} \right)^2 \left[-\frac{2}{z^3} \ln \left(\frac{z_{1,1}^{k,m} - z}{z_{1,0}^{k,m} - z} \right) \frac{z_{1,0}^{k,m}}{z_{1,1}^{k,m}} + \frac{1}{z^2} \left(\frac{1}{z_{1,0}^{k,m} - z} - \frac{1}{z_{1,1}^{k,m} - z} \right) \right. \\
 &\left. - \frac{1}{z^2} \left(\frac{1}{z_{1,1}^{k,m}} - \frac{1}{z_{1,0}^{k,m}} \right) \right] + \frac{1}{2} \left(\frac{\overline{z_{0,1}^{k,m}}}{z_{0,1}^{k,m}} \right)^2 \\
 &\times \left[z_{0,1}^{k,m} - z_{1,1}^{k,m} + 2z \ln \left(\frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} \right) + z^2 \left(\frac{1}{z_{1,1}^{k,m} - z} - \frac{1}{z_{0,1}^{k,m} - z} \right) \right] \\
 &+ \frac{1}{2} \left(\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}} \right)^2 \left[-\frac{2}{z^3} \ln \left(\frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z} \right) \frac{z_{0,1}^{k,m}}{z_{0,0}^{k,m}} + \frac{1}{z^2} \left(\frac{1}{z_{0,1}^{k,m} - z} - \frac{1}{z_{0,0}^{k,m} - z} \right) \right. \\
 &\left. - \frac{1}{z^2} \left(\frac{1}{z_{0,0}^{k,m}} - \frac{1}{z_{0,1}^{k,m}} \right) \right], \tag{20}
 \end{aligned}$$

$$\begin{aligned}
 S_{k,m}^c(z) &= \left(\left(\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}} \right) - \frac{1}{3} \left(\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}} \right)^3 \right) \left[\frac{(z_{1,0}^{k,m})^2}{2} - \frac{(z_{0,0}^{k,m})^2}{2} \right. \\
 &+ 2z \left(z_{1,0}^{k,m} - z_{0,0}^{k,m} \right) + 3z^2 \ln \frac{z_{1,0}^{k,m} - z}{z_{0,0}^{k,m} - z} + z^3 \left(\frac{1}{z_{0,0}^{k,m} - z} - \frac{1}{z_{1,0}^{k,m} - z} \right) \left. \right] \\
 &+ \left(z_{1,0}^{k,m} \overline{z_{1,0}^{k,m}} \right) \left\{ \ln \left(\frac{z_{1,0}^{k,m} - z}{z_{1,0}^{k,m} - z} \right) + z \left(\frac{1}{z_{1,0}^{k,m} - z} - \frac{1}{z_{1,1}^{k,m} - z} \right) \right\} - \frac{(z_{1,0}^{k,m} \overline{z_{1,0}^{k,m}})^3}{3} \\
 &\times \left\{ \frac{3}{z^4} \cdot \ln \left(\frac{z_{1,1}^{k,m}}{z_{1,0}^{k,m}} \right) - \frac{2}{z^3} \cdot \left(\frac{1}{z_{1,1}^{k,m}} - \frac{1}{z_{1,0}^{k,m}} \right) - \frac{1}{2z^2} \cdot \left(\frac{1}{(z_{1,1}^{k,m})^2} - \frac{1}{(z_{1,0}^{k,m})^2} \right) \right. \\
 &\left. - \frac{3}{z^4} \cdot \ln \left(\frac{z_{1,1}^{k,m} - z}{z_{1,0}^{k,m} - z} \right) + \frac{1}{z^3} \cdot \left(\frac{1}{z_{1,0}^{k,m} - z} - \frac{1}{z_{1,1}^{k,m} - z} \right) \right\} \\
 &+ \left(\left(\frac{\overline{z_{0,1}^{k,m}}}{z_{0,1}^{k,m}} \right) - \frac{1}{3} \left(\frac{\overline{z_{0,1}^{k,m}}}{z_{0,1}^{k,m}} \right)^3 \right) \left[\frac{(z_{0,1}^{k,m})^2}{2} - \frac{(z_{1,1}^{k,m})^2}{2} + 2z \left(z_{0,1}^{k,m} - z_{1,1}^{k,m} \right) \right. \\
 &+ 3z^2 \ln \left(\frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} \right) + z^3 \left(\frac{1}{z_{1,1}^{k,m} - z} - \frac{1}{z_{0,1}^{k,m} - z} \right) \left. \right] \\
 &+ \left(z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}} \right) \left\{ \ln \left(\frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z} \right) + z \left(\frac{1}{z_{0,1}^{k,m} - z} - \frac{1}{z_{0,0}^{k,m} - z} \right) \right\} - \frac{(z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}})^3}{3} \\
 &\times \left\{ \frac{3}{z^4} \cdot \ln \left(\frac{z_{0,0}^{k,m}}{z_{0,1}^{k,m}} \right) - \frac{2}{z^3} \cdot \left(\frac{1}{z_{0,0}^{k,m}} - \frac{1}{z_{0,1}^{k,m}} \right) - \frac{1}{2z^2} \cdot \left(\frac{1}{(z_{0,0}^{k,m})^2} - \frac{1}{(z_{0,1}^{k,m})^2} \right) \right. \\
 &\left. - \frac{3}{z^4} \cdot \ln \left(\frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z} \right) + \frac{1}{z^3} \cdot \left(\frac{1}{z_{0,1}^{k,m} - z} - \frac{1}{z_{0,0}^{k,m} - z} \right) \right\}, \tag{21}
 \end{aligned}$$

$$\begin{aligned}
S_{k,m}^d(z) &= \frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}} \left[\ln \frac{z_{1,0}^{k,m} - z}{z_{0,0}^{k,m} - z} + z \left(\frac{1}{z_{0,0}^{k,m} - z} - \frac{1}{z_{1,0}^{k,m} - z} \right) \right] \\
&- z_{1,1}^{k,m} \overline{z_{1,1}^{k,m}} \frac{1}{z^2} \ln \left(\frac{z_{1,1}^{k,m} - z}{z_{1,0}^{k,m} - z} \right) \frac{z_{1,0}^{k,m}}{z_{1,1}^{k,m}} + z_{1,1}^{k,m} \overline{z_{1,1}^{k,m}} \frac{1}{z} \left(\frac{1}{z_{1,0}^{k,m} - z} - \frac{1}{z_{1,1}^{k,m} - z} \right) \\
&+ \frac{\overline{z_{1,1}^{k,m}}}{z_{1,1}^{k,m}} \left[\ln \frac{z_{0,1}^{k,m} - z}{z_{1,1}^{k,m} - z} + z \left(\frac{1}{z_{1,1}^{k,m} - z} - \frac{1}{z_{0,1}^{k,m} - z} \right) \right] \\
&- \frac{\overline{z_{0,0}^{k,m}} z_{0,0}^{k,m}}{z^2} \ln \left(\frac{z_{0,0}^{k,m} - z}{z_{0,1}^{k,m} - z} \right) \frac{z_{0,1}^{k,m}}{z_{0,0}^{k,m}} + \frac{z_{0,0}^{k,m} \overline{z_{0,0}^{k,m}}}{z} \left(\frac{1}{z_{0,1}^{k,m} - z} - \frac{1}{z_{0,0}^{k,m} - z} \right).
\end{aligned} \tag{22}$$

Therefore, we obtain quadrature formula

$$\begin{aligned}
S(\rho|z) &\approx \sum_{k=1}^{N-1} \sum_{m=0}^{M_k-1} \left(a_{k,m} S_{k,m}^a(z) + b_{k,m} S_{k,m}^b(z) \right. \\
&\quad \left. + c_{k,m} S_{k,m}^c(z) + d_{k,m} S_{k,m}^d(z) \right), z \in \Omega^{(2)},
\end{aligned}$$

where coefficients $a_{k,m}$, $b_{k,m}$, $c_{k,m}$, and $d_{k,m}$ and integrals $S_{k,m}^a$, $S_{k,m}^b$, $S_{k,m}^c$ and $S_{k,m}^d$ are defined by (5), (6), (7), (8) and (19), (20), (21), (22), respectively.

5 Error analysis

In this section, we present numerical results of error for proposed approximate formulas of T and S in test examples by using MATLAB program.

Let us take $l_2(\Omega^{(j)})$ ($j = 1, 2$), the spaces of the grid functions $\rho^{\tau,h}$, defined on $\Omega^{(j)}$ equipped with the norm

$$\left\| \rho^{\tau,h} \right\|_{l_2(\Omega^{(j)})} = \sqrt{\pi\tau} \left[\sum_{k=1}^{N-1} \sum_{m=1}^{M_k-1} |\rho(r_k e^{i\theta_{k,m}})|^2 \right]^{1/2}, \tag{23}$$

where $\rho_{k,m} = \rho(r_k e^{i\theta_{k,m}})$, $r_k e^{i\theta_{k,m}} \in \Omega^{(j)}$.

Table 1 displays the value of error of $T(\rho|z)$ defined by

$$ErT = \sqrt{\pi\tau} \left[\sum_{k=1}^{N-1} \sum_{m=1}^{M_k-1} |T(\rho|z_{0,0}^{*k,m}) - T(\widehat{\rho}|z_{0,0}^{*k,m})|^2 \right]^{1/2} \tag{24}$$

for $N = 5, 10, 20, 40$.

Table 1. Error analysis for $T(\rho|z)$

	$\rho(z)$	$T(\rho z)$	$N = 5$	$N = 10$	$N = 20$	$N = 40$
1	$z\bar{z}$	$\frac{z\bar{z}^2}{2}$	0.2450	0.0278	0.0059	0.0014
2	$z\bar{z}^2$	$\frac{z\bar{z}^3}{3}$	0.1803	0.0162	0.0038	9.4×10^{-04}
3	$z\bar{z}^3$	$\frac{z\bar{z}^4}{4}$	0.1611	0.0338	0.0076	0.0019
4	z	$z\bar{z} - 1$	0.3018	0.0377	0.0047	5.8×10^{-04}
5	\bar{z}	$\frac{\bar{z}^2}{2}$	0.0989	0.0124	0.0015	1.9×10^{-04}
6	z^2	$z^2\bar{z} - z$	0.1902	0.0227	0.0048	0.0012
7	\bar{z}^2	$\frac{\bar{z}^3}{3}$	0.1489	0.0206	0.0047	0.0012
8	\bar{z}^3	$\frac{\bar{z}^4}{4}$	0.2027	0.0526	0.0133	0.0033
9	$z^2\bar{z}$	$\frac{z^2\bar{z}^2}{2} - \frac{1}{2}$	0.1112	0.0226	0.0059	0.0015
10	$z^2\bar{z}^2$	$\frac{z^2\bar{z}^3}{3}$	0.1417	0.0194	0.0048	0.0012
11	$z^2\bar{z}^3$	$\frac{z^2\bar{z}^4}{4}$	0.1215	0.0212	0.0049	0.0012
12	$z^3\bar{z}$	$\frac{z^3\bar{z}^2}{2} - \frac{z}{2}$	0.1369	0.0279	0.0061	0.0015
13	$z^3\bar{z}^2$	$\frac{z^3\bar{z}^3}{3} - \frac{1}{3}$	0.1144	0.0344	0.0086	0.0022
14	$z^3\bar{z}^3$	$\frac{z^3\bar{z}^4}{4}$	0.0885	0.0181	0.0045	0.0011
15	$z^3\bar{z}^4$	$\frac{z^3\bar{z}^5}{5}$	0.1096	0.0254	0.0057	0.0014
16	z^3	$z^3\bar{z} - z^2$	0.1807	0.0388	0.0096	0.0024
17	$z^2 - \bar{z}^2$	$z^2\bar{z} - z - \frac{\bar{z}^3}{3}$	0.1785	0.0112	6.9×10^{-04}	4.3×10^{-05}

Table 2 layouts the value of error of $S(\rho|z)$ defined by

$$ErS = \sqrt{\pi\tau} \left[\sum_{k=1}^{N-1} \sum_{m=1}^{M_k-1} |S(\rho|z_{0,0}^{*k,m}) - S(\hat{\rho}|z_{0,0}^{*k,m})|^2 \right]^{1/2} \tag{25}$$

for $N = 5, 10, 20, 40$.

6 Conclusion

In this paper, we construct second-order of accuracy quadrature formulas for the numerical calculation of the Vekua types two-dimensional potential and singular integral operators in the unit disk of complex plane. We propose quadrature formulas for these integrals which based on first-order spline approximation of two-dimensional function. MATLAB program are used for numerical experiments in test examples.

Table 2. Error analysis for $S(\rho|z)$

	$\rho(z)$	$S(\rho z)$	$N = 5$	$N = 10$	$N = 20$	$N = 40$
1	$z\bar{z}$	$\frac{\bar{z}^2}{2}$	0.0775	0.0138	0.0030	7.1×10^{-04}
2	$z\bar{z}^2$	$\frac{\bar{z}^3}{3}$	0.4025	0.0683	0.0230	0.0083
3	$z\bar{z}^3$	$\frac{\bar{z}^4}{4}$	0.6517	0.1549	0.0567	0.0207
4	z	\bar{z}	8.7×10^{-15}	1.5×10^{-13}	2.1×10^{-12}	5.1×10^{-11}
5	\bar{z}	0	0.3072	0.0768	0.0192	0.0048
6	z^2	$2z\bar{z} - 1$	0.6155	0.1169	0.0324	0.0110
7	\bar{z}^2	0	0.6525	0.1174	0.0322	0.0110
8	\bar{z}^3	0	0.7794	0.1901	0.0685	0.0249
9	$z^2\bar{z}$	$z\bar{z}^2$	0.4335	0.0724	0.0239	0.0085
10	$z^2\bar{z}^2$	$\frac{2}{3}z\bar{z}^3$	0.0985	0.0261	0.0067	0.0017
11	$z^2\bar{z}^3$	$\frac{1}{2}z\bar{z}^4$	0.3466	0.0906	0.0332	0.0121
12	$z^3\bar{z}$	$\frac{3}{2}z^2\bar{z}^2 - \frac{1}{2}$	0.5976	0.1478	0.0557	0.0206
13	$z^3\bar{z}^2$	$z^2\bar{z}^3$	0.3912	0.0981	0.0346	0.0124
14	$z^3\bar{z}^3$	$\frac{3}{4}z^2\bar{z}^4$	0.1748	0.0463	0.0118	0.0030
15	$z^3\bar{z}^4$	$\frac{3}{5}z^2\bar{z}^5$	0.3654	0.1114	0.0407	0.0150
16	z^3	$3z^2\bar{z} - 2z$	0.8176	0.1897	0.0682	0.0248
17	$z^2 - \bar{z}^2$	$2z\bar{z} - 1$	0.7305	0.0913	0.0114	0.0014

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