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Well-posedness of boundary value problems for reverse parabolic equation with integral condition

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Abstract. Reverse parabolic equation with integral condition is considered. Well-posedness of reverse parabolic problem in the Hölder space is proved. Coercive stability estimates for solution of three boundary value problems (BVPs) to reverse parabolic equation with integral condition are established.

Keywords. Reverse parabolic problem, stability, coercive stability, well-posedness.

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1 Introduction

Well-posedness of nonclassical BVPs for parabolic differential equations has been studied extensively by many researchers (see, e.g., [1–11, 14, 15] and bibliography therein).

Let H be Hilbert space and $A : H \to H$ be self-adjoint positive definite (SAPD) operator such that $A > \delta I$ for identity operator $I : H \to H$ and some positive number δ. In the paper [5], well-posedness of the reverse parabolic problem

$$\frac{du(t)}{dt} - Au(t) = f(t), 0 \leq t \leq 1, \tag{1}$$

with multipoint nonlocal condition

$$\begin{aligned} &u(1) = \sum_{k=1}^{p} \alpha_k u(\theta_k) + \varphi, \\ &0 \leq \theta_1 < \theta_2 < ... < \theta_p < 1 \end{aligned} \tag{2}$$

was established in in the space of smooth functions. In applications, coercivity estimates for the solution of parabolic differential equations were obtained. Well-posedness of problem (1), (2) was established under the assumption

$$|\alpha_k| \leq 1. \tag{3}$$

In the papers [6, 8], stable finite difference schemes for the approximate solution of the reverse multidimensional parabolic differential equation with various multi-point boundary conditions are proposed. Coercive stability estimates for difference schemes are obtained. In [1, 7, 9–11, 14, 15], differential and difference problems of determining the parameter in a parabolic equations were studied.

In this work, we study reverse problem for parabolic equation (1) with integral type nonlocal condition:

$$u(1) = \int_0^1 \rho(s)u(s)ds + \varphi. \tag{4}$$

Suppose that a continuous real valued scalar function ρ be under assumption:

$$\int_0^1 |\rho(s)|\, ds \leq 1. \tag{5}$$

A function $u : [0, 1] \to H$ is said to be a solution of the problem (1), (4) if the following three conditions are valid:

1. $u(t)$ is continuously differentiable on $[0, 1]$.
2. For $\forall\, t \in [0, 1]$ the element $u(t)$ belongs to $D(A)$ and the function $Au(t)$ is continuous on $[0, 1]$.
3. $u(t)$ satisfies the equation (1) and the nonlocal condition (4).

Denote by $C(H)$ and $C_1^\alpha(H)$, the Banach space of all continuous functions $v : [0, 1] \to H$ equipped with the suitable norms

$$\|v\|_{C(H)} = \max_{0\leq t\leq 1} \|v(t)\|_H ,$$

$$\|v\|_{C_1^\alpha(H)} = \|v\|_{C(H)} + \sup_{0\leq t<t+\tau\leq 1} \left(\tfrac{1-t}{\tau}\right)^\alpha \|v(t+\tau) - v(t)\|_H .$$

Lemma 1.1. [3] *For every* $0 < t < t + \tau \leq 1$ *and* $0 \leq \beta \leq 1$, *the following inequalities*

$$\begin{gathered} \left\|e^{-tA}\right\|_{H\to H} \leq 1, \ \left\|tAe^{-tA}\right\|_{H\to H} \leq 1, \\ \left\|e^{-tA} - e^{-(t+\tau)A}\right\|_{H\to H} \leq M\frac{\tau^\beta}{(t+\tau)^\beta}, \\ \left\|A\left(e^{-tA} - e^{-(t+\tau)A}\right)\right\|_{H\to H} \leq M\frac{\tau^\beta}{t(t+\tau)^\beta} \end{gathered} \tag{6}$$

are fulfilled for some positive M.

Lemma 1.2. *Suppose that the assumption (5) holds. Then, the operator*

$$S = I - \int_0^1 \rho(\tau) e^{-(1-\tau)A} d\tau \ = I - D$$

has a bounded inverse $Q = S^{-1}$ *such that*

$$\|Q\|_{H \to H} \leq M. \tag{7}$$

Proof. By using spectral representation [13] for A and Cauchy inequality, we have

$$\begin{aligned}
&\left\langle e^{-(1-\tau)A}u, u\right\rangle \leq \left\|e^{-(1-\tau)A}u\right\|_H \cdot \|u\|_H \\
&\leq \left\|e^{-(1-\tau)A}u\right\|_{H\to H} \cdot \|u\|_H \cdot \|u\|_H \\
&\leq \sup_{\delta\leq\mu<\infty} \left|e^{-(1-\tau)\mu}\right| \cdot \|u\|_H^2 \leq e^{-(1-\tau)\delta} \langle u, u\rangle \\
&\leq \langle u, u\rangle .
\end{aligned}$$

Thus, from (5) it follows that

$$\begin{aligned}
&\langle (I-D)u, u\rangle = \langle u, u\rangle - \langle Du, u\rangle \\
&\geq \langle u, u\rangle - \int_0^1 |\rho(\tau)| \ d\tau \ \langle u, u\rangle \\
&= (1 - \int_0^1 |\rho(\tau)| \ d\tau) \langle u, u\rangle \\
&> \rho_0 \langle u, u\rangle .
\end{aligned}$$

Here $\rho_0 > 0$.So, there exists bounded inverse Q and inequality (7) holds. □

2 Well-posedness of reverse parabolic problem

Theorem 2.1. *Assume that (5) is valid,* $\varphi \in D(A)$, $f(t) \in C_1^\alpha(H)$. *Then, problem (1), (4) has unique solution and it is well-possed in* $C_1^\alpha(H)$ *and the coercive estimate*

$$\left\|u'\right\|_{C_1^\alpha(H)} + \|Au\|_{C_1^\alpha(H)} \leq M(\delta) \left(\|A\varphi\|_H + \frac{1}{\alpha(1-\alpha)} \|f\|_{C_1^\alpha(H)}\right)$$

is fulfilled, where $M(\delta)$ *is independent of* φ *and* f.

Proof. If $u(1)$ is given, then the solution of parabolic equation (1) is defined by

$$u(t) = e^{-(1-t)A}u(1) - \int_t^1 e^{-(s-t)A} f(s) ds \ . \tag{8}$$

By using (8) and nonlocal condition (4), we have

$$u(1)=\int_0^1\rho(\tau)e^{-(1-\tau)A}d\tau\cdot u(1)-\int_0^1\rho(\tau)\int_t^1 e^{-(s-\tau)A}f(s)dsd\tau+\varphi.$$

By Lemma (2.1), we can obtain

$$u(1)=Q\left(-\int_0^1\int_t^1\rho(\tau)e^{-(s-\tau)A}f(s)dsd\tau+\varphi\right). \tag{9}$$

Hence, reverse problem (1), (4) has unique solution which is derived by formulas (8) and (9). We have

$$\begin{aligned}&Au(t)=e^{-(1-t)A}Au(1)-\int_t^1 Ae^{-(s-t)A}f(s)ds\\&=e^{-(1-t)A}Au(1)-\int_t^1 Ae^{-(s-t)A}\left[f(s)-f(t)\right]ds\\&+\left(e^{-(1-t)A}-I\right)f(t)\end{aligned} \tag{10}$$

for any $t\in(0,1)$. Applying definition of $C_1^\alpha(H)$-norm and corresponding estimates of Lemma (2.1), we have

$$\begin{aligned}&\|Au(t)\|_H\le\|Au(1)\|_H+\int_t^1\frac{\|f\|_{C_1^\alpha(H)}}{(1-t)^\alpha(s-t)^{1-\alpha}}ds+(1+1)\|f\|_{C_1^\alpha(H)}\\&\le\|Au(1)\|_H+\left(\frac{1}{\alpha}+2\right)\|f\|_{C_1^\alpha(H)}.\end{aligned} \tag{11}$$

From (9), it can be obtained

$$\begin{aligned}Au(1)&=Q\left(-\int_0^1\rho(\tau)\int_t^1 Ae^{-(s-\tau)A}f(s)dsd\tau+A\varphi\right)\\&=Q\left\{-\int_0^1\rho(\tau)\left[\int_t^1 Ae^{-(s-\tau)A}\left(f(s)-f(\tau)\right)ds\right]d\tau\right.\\&\left.+\int_0^1\rho(\tau)\left[\left(e^{-(1-\tau)A}-e^{-(t-\tau)A}\right)f(\tau)\right]d\tau+A\varphi\right\}.\end{aligned} \tag{12}$$

Then, by using assumption (4), Lemmas (2.1) and (2.2), the definition of $C_1^\alpha(H)$-norm, we obtain

$$\|Au(1)\|_H \leq M \left\{ \int_0^1 |\rho(\tau)| \left[\int_t^1 \frac{\|f\|_{C_1^\alpha(H)}}{(1-t)^\alpha (s-t)^{1-\alpha}} ds \right] d\tau \right. \left. +2 \|f\|_{C_1^\alpha(H)} \int_0^1 |\rho(\tau)| \, d\tau + \|A\varphi\|_H \right\} \leq M \left(\frac{1}{\alpha} \|f\|_{C_1^\alpha(H)} + \|A\varphi\|_H \right). \tag{13}$$

Combining (11) and (13), we have

$$\|Au\|_{C(H)} \leq M \left(\frac{1}{\alpha} \|f\|_{C_1^\alpha(H)} + \|A\varphi\|_H \right). \tag{14}$$

Now, let us estimate

$$\sup_{0 \leq t < t+\tau \leq 1} \left(\frac{1-t}{\tau}\right)^\alpha \|Au(t+\tau) - Au(t)\|_H .$$

There are two cases. First case is $1 - t \leq 2\tau$. From (14), it follows that

$$\|Au(t+\tau) - Au(t)\|_H \leq M \left(\frac{\tau}{1-t}\right)^\alpha \left(\frac{1}{\alpha} \|f\|_{C_1^\alpha(H)} + \|A\varphi\|_H \right). \tag{15}$$

Second case is $1 - t > 2\tau$. Identity (10) yields

$$Au(t+\tau) - Au(t) = \sum_{i=1}^{5} K_i(t),$$

where

$$K_1(t) = \left(e^{-(1-t-\tau)A} - e^{-(1-t)A}\right) Au(1),$$

$$K_2(t) = \int_{t+2\tau}^{1} A \left[e^{-(s-t)A} - e^{-(s-t-\tau)A}\right] [f(s) - f(t)] \, ds,$$

$$K_3(t) = \int_{t}^{t+2\tau} Ae^{-(s-t)A} [f(s) - f(t)] \, ds,$$

$$K_4(t) = -\int_{t+\tau}^{t+2\tau} Ae^{-(s-t-\tau)A} [f(s) - f(t+\tau)] \, ds,$$

$$K_5(t) = \left[e^{-(1-t-\tau)A} - I\right] f(t+\tau) - \left[e^{-(1-t)A} - I\right] f(t) + \left[e^{-(1-t-\tau)A} - e^{-\tau A}\right] (f(t+\tau) - f(t)).$$

By using Lemma (2.1), inequality (13) and the definition of $C_1^\alpha(H)$-norm, it can be showed that

$$\|K_1(t)\|_H \leq \left\|e^{-(1-t-\tau)A} - e^{-(1-t)A}\right\|_{H\to H} \|Au(1)\|_H$$
$$\leq M\frac{\tau^\alpha}{(1-t)^\alpha}\left(\frac{1}{\alpha}\|f\|_{C_1^\alpha(H)} + \|A\varphi\|_H\right)$$

and

$$\|K_2(t)\|_H \leq \int\limits_{t+2\tau}^{1} \left\|A\left[e^{-(s-t)A} - e^{-(s-t-\tau)A}\right]\right\|_{H\to H} \|f(s)-f(t)\|_H\,ds$$
$$\leq \frac{2^{-1+\alpha}\tau^\alpha}{(1-t)^\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)}.$$

In the similar manner, we have

$$\|K_3(t)\|_H \leq \frac{2^\alpha\tau^\alpha}{(1-t)^\alpha\alpha}\|f\|_{C_1^\alpha(H)},$$

$$\|K_4(t)\|_H \leq \frac{2^\alpha\tau^\alpha}{(1-t)^\alpha\alpha}\|f\|_{C_1^\alpha(H)}.$$

By using triangle inequality and the definition of corresponding norms, we get

$$\|K_5(t)\|_H \leq 2\ \|f(t+\tau)\|_H + 2\|f(t)\|_H + 2\|f(t+\tau)\|_H + 2\|f(t)\|_H$$
$$\leq 8\|f\|_{C(H)} \leq M\|f\|_{C_1^\alpha(H)}.$$

Combining estimates for $\|K_i(t)\|_H$, $i = 1,2,3,4,5,$ we have

$$\frac{(1-t)^\alpha\|Au(t+\tau)-Au(t)\|_H}{\tau^\alpha} \leq M\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)} + \|A\varphi\|_H\right).$$

Thus,

$$\sup_{0\leq t<t+\tau\leq 1}\left(\frac{1-t}{\tau}\right)^\alpha\|Au(t+\tau)-Au(t)\|_H$$
$$\leq M\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)} + \|A\varphi\|_H\right).$$

Therefore,

$$\|Au\|_{C_1^\alpha(H)} \leq M\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)} + \|A\varphi\|_H\right). \tag{16}$$

Finally, by using equation (1), triangle inequality and estimate (16), we get

$$\|u'\|_{C_1^\alpha(H)} \leq M\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)} + \|A\varphi\|_H\right). \tag{17}$$

□

3 Applications to BVPs

Now, we will apply abstract results of previous section to study on well-posedness of three BVPs.

Let us σ is a given positive number, and

$$a:(0,1)\to R,\varphi:[0,1]\to R,f:(0,1)\times(0,1)\to R$$

are given smooth functions and $\varphi\in L_2[0,1]$. Moreover, $\forall x\in\mathbf{\Omega},\ a(x)\geq a_0>0$.

First, we consider boundary value problem for one dimensional parabolic equation with integral type nonlocal condition

$$\begin{cases} u_t(x,t)+\ (a(x)u_x(x,t))_x-\sigma u(x,t)=f(x,t), \\ 0<t<1,0<x<1, \\ u(x,1)=\int\limits_0^1\rho(s)u(x,s)ds+\varphi(x),\ x\in\ [0,1], \\ u_x(0,t)=u_x(1,t),\ u(0,t)=u(1,t),\ 0\leq t\leq 1. \end{cases} \tag{18}$$

Notice that the differential expression

$$A^x v=-(a(x)v_x(x))_x+\sigma v(x) \tag{19}$$

defines SAPD operator A^x with domain

$$D(A^x)=\{v,v_x,v_{xx}\in L_2[0,1]:v_x(0)=v_x(1),\ v(0)=v(1)\}.$$

This allows us to reduce the nonlocal BVP (18) to the nonlocal BVP (1), (4) in a Hilbert space $H=L_2[0,1]$ with a SAPD operator A^x derived by (19). Therefore, it can be formulated the following statement on well-posedness of reverse problem (18).

Theorem 3.1. *Suppose that $\varphi\in W_2^2(\overline{\mathbf{\Omega}}), f\in C_1^\alpha(L_2[0,1])$ and assumption (5) is valid. Then, for solution of BVP (18) the following stability estimate*

$$\begin{aligned} &\|u_t\|_{C_1^\alpha(L_2[0,1])}+\|u\|_{C_1^\alpha(W_2^2([0,1]))} \\ \leq\ &M\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(L_2[0,1])}+\|\varphi\|_{W_2^2([0,1])}\right) \end{aligned} \tag{20}$$

holds, where the constant M does not depend on f and φ.

Let $\Omega=(0,1)^n\subset R^n$ is unit open cube with boundary S, $\overline{\mathbf{\Omega}}=S\cup\mathbf{\Omega}$ and

$$a_r:\mathbf{\Omega}\to R,\varphi:\overline{\mathbf{\Omega}}\to R,f:(0,1)\times\mathbf{\Omega}\to R$$

are given smooth functions. Moreover, $\forall x\in\mathbf{\Omega},\ a_r(x)\geq a_0>0$, σ is given positive number.

Denote by $L_2(\overline{\Omega})$ and $W_2^2(\overline{\Omega})$ the Hilbert spaces of all integrable functions $v(x)$, defined on $\overline{\Omega}$, equipped with the corresponding norms

$$\|v\|_{L_2(\overline{\Omega})}=\left\{\int\limits_{x\in\overline{\Omega}}|v(x)|^2\,dx_1\dots dx_n\right\}^{\frac{1}{2}},$$
$$\|v\|_{W_2^2(\overline{\Omega})}=\left\{\int\limits_{x\in\overline{\Omega}}\left(|v(x)|^2+\sum_{i=1}^{n}\sum_{j=1}^{n}\left|v_{x_ix_j}(x)\right|^2\right)dx_1\dots dx_n\right\}^{\frac{1}{2}}.$$

Second, we consider BVP for multidimensional parabolic equation with Dirichlet boundary condition

$$\begin{cases} u_t(x,t)+\sum\limits_{r=1}^{n}\left(a_r(x)u_{x_r}(x,t)\right)_{x_r}-\sigma u(x,t)=f(x,t),\\ x=(x_1,x_2,...,x_n)\in\mathbf{\Omega},0<t<1,\\ u(x,1)=\int\limits_0^1\rho(s)u(x,s)ds+\varphi(x),\ x\in\overline{\Omega},\\ u(x,t)=0,\ x\in S,0\leq t\leq 1. \end{cases} \tag{21}$$

Denote by

$$A^xv=-\sum_{r=1}^{n}(a_r(x)v_{x_r})_{x_r}+\sigma v \tag{22}$$

differential expression of multidimensional parabolic equation of (21). It defines a SAPD operator A^x acting on $L_2(\overline{\Omega})$ with the domain [13]

$$D(A^x)=\left\{v(x)\in W_2^2(\overline{\Omega}),\ v=0\text{ on }S\right\}.$$

So, from abstract Theorem 2.1 it can be concluded statement on well-posedness of multidimensional reverse parabolic problem (21).

Theorem 3.2. *Let $\varphi\in W_2^2(\overline{\Omega})$, $f\in C_1^\alpha(L_2(\overline{\Omega}))$ and suppose that assumption (5) is valid. Then, for solution of multidimensional BVP (21) the following stability estimate*

$$\|u_t\|_{C_1^\alpha(L_2(\overline{\Omega}))}+\|u\|_{C_1^\alpha(W_2^2(\overline{\Omega}))}\leq M\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(L_2(\overline{\Omega}))}+\|\varphi\|_{W_2^2(\overline{\Omega})}\right) \tag{23}$$

is fulfilled, where the constant M does not depend on f and φ.

Third, we consider BVP for multidimensional parabolic equation with Neumann boundary condition

$$\begin{cases} u_t(x,t) + \sum\limits_{r=1}^{n} (a_r(x)u_{x_r}(x,t))_{x_r} - \sigma u(x,t) = f(x,t), \\ x = (x_1, x_2, ..., x_n) \in \Omega, 0 < t < 1; \\ u(x,1) = \int\limits_0^1 \rho(s)u(x,s)ds + \varphi(x),\ x \in \overline{\Omega}, \\ \frac{\partial u(x,t)}{\partial \overrightarrow{n}} = 0,\ x \in S, 0 \le t \le 1. \end{cases} \tag{24}$$

Differential expression (22) defines a SAPD operator A^x acting on $L_2(\overline{\Omega})$ with the domain $D(A^x)$=$\left\{u(x) \in W_2^2(\overline{\Omega}),\ u = 0 \text{ on } S\right\}$ ([13]). Therefore, abstract Theorem 2.1 implies the well-posedness of reverse parabolic problem (24).

Theorem 3.3. *Suppose that $\varphi \in W_2^2(\overline{\Omega}), f \in C_1^\alpha(L_2(\overline{\Omega}))$ and assumption (5) is valid. Then, for solution of multidimensional BVP (24) the stability estimate (23) is valid, where the constant M is independent from f and φ.*

4 Conclusion

In the present paper, we discuss stability estimates for the solution of reverse parabolic problem with integral condition. Abstract results are applied to three BVPs for multidimensional parabolic differential equation with integral boundary condition. Theorems on well-posedness of these BVPs are presented.

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