Contemporary Analysis and Applied Mathematics
Vol. 4, No. 1, 107-121, 2016
DOI: 10.18532/caam.64780

A third order of accuracy difference scheme for the Neumann type overdetermined elliptic problem

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Received: 24 January 2016 Accepted: 3 March 2016

Abstract. In this paper, a third order of accuracy difference scheme for the approximation of the solution of elliptic identification problem with Neumann type overdetermination is presented. We obtain stability estimates for the solutions of constructed difference scheme. Furthermore, a third order of accuracy difference scheme for Neumann type overdetermined multidimensional elliptic problem with Dirichlet boundary condition is constructed. Finally, a numerical example for two-dimensional problem is given.

Key words. Difference scheme, identification problem, inverse elliptic problem, stability, coercive stability.

1 Introduction

Identification problems for elliptic type differential and difference equations and their applications were studied extensively by several researchers (see [1–10] and the references therein). Dirichlet type overdetermined inverse problems for elliptic differential equations and their approximations were investigated in [4–8]. Particularly, papers [7,8] are devoted to construct high order of accuracy stable difference schemes for inverse problem with Dirichlet type overdetermination.

Stable first and second order of accuracy difference schemes (ADS) for the following Neumann type overdetermined elliptic problem with a self-adjoint positive definite operator A in an arbitrary Hilbert space H to find a function u and an element $p \in H$

$$\begin{cases} -u_{tt}(t) + Au(t) = f(t) + pt, t \in (0,1), \\ u_t(0) = \varphi, \ u_t(\lambda) = \xi, u_t(1) = \psi, \ 0 < \lambda < 1 \end{cases} \tag{1.1}$$

are presented in [8].

We also apply the abstract results to approximate the following Neumann type overdetermined inverse problem for the multi-dimensional elliptic equation with Dirichlet boundary condition

$$\begin{cases} -u_{tt}(t,x) - \sum\limits_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \sigma u(t,x) = f(t,x) + p(x)t, \\ x = (x_1, \cdots, x_n) \in \Omega, \ t \in (0,1), \\ u(0,x) = \varphi(x), u(1,x) = \psi(x), u(\lambda,x) = \xi(x), x \in \overline{\Omega}, \\ u(t,x) = 0, \ x \in S, \ t \in [0,1]. \end{cases} \tag{1.2}$$

Here, $\Omega = (0,1) \times \cdots \times (0,1)$ is the open cube in Euclidean space R^n with boundary S, $\overline{\Omega} = \Omega \cup S$, $a_r(x)$ $(x \in \Omega)$, $\varphi(x), \xi(x), \psi(x)$ $(x \in \overline{\Omega})$, $f(t,x)$ $(t \in (0,1), \ x \in \Omega)$ are known smooth functions, $a_r(x) \geq a > 0 (x \in \overline{\Omega})$, and $0 < \lambda < 1$, $\sigma > 0$ are given numbers.

We would like point out that the first and second order of ADS for (1.2) were presented in [8]. The paper [9] is devoted to numerical solution of Neumann type overdetermined inverse elliptic problem with mixed boundary conditions. In the present paper, we construct a third order of ADS for problem (1.2) and establish stability estimates for its solutions.

The paper is organized as follows. Section 2 describes a third order of ADS for inverse problem (1.1) and here stability estimates for its solution are given. In Section 3, we construct a third order of accuracy stable difference scheme for problem (1.2). Then, the results of numerical experiments are displayed in Section 4. Finally, we give concluding remarks in Section 5.

2 Difference scheme for identification problem

Let N be a given natural number and $N\tau = 1$. To present a third order of ADS for approximation aforementioned problem (1.1), let us introduce the set of grid points $\{t_k = k\tau, \ 0 \leq k \leq N \}$ and denote by $C_\tau(H), \mathcal{C}_\tau^\alpha(H)$, and $\mathcal{C}_\tau^{\alpha,\alpha}(H)$ $(0 < \alpha < 1)$ the spaces of H-valued grid functions $\{q_k\}_{k=1}^{N-1}$ with the following norms

$$\left\| \{q_k\}_{k=1}^{N-1} \right\|_{\mathcal{C}_\tau(H)} = \max_{1 \leq k \leq N-1} \|q_k\|_H \,,$$

$$\left\| \{q_k\}_{k=1}^{N-1} \right\|_{\mathcal{C}_\tau^\alpha(H)} = \left\| \{q_k\}_{k=1}^{N-1} \right\|_{\mathcal{C}_\tau(H)} + \sup_{1\le k<k+n\le N-1} \frac{\|q_{k+n}-q_k\|_H}{(n\tau)^\alpha},$$

$$\begin{aligned}\left\| \{q_k\}_{k=1}^{N-1} \right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)} &= \left\| \{q_k\}_{k=1}^{N-1} \right\|_{\mathcal{C}_\tau(H)} \\ &+ \sup_{1\le k<k+n\le N-1} \frac{(k\tau+n\tau)^\alpha(1-k\tau)^\alpha\|q_{k+n}-q_k\|_H}{(n\tau)^\alpha},\end{aligned}$$

respectively. Let I be the identity operator and $C = A + \frac{\tau^2}{12}A^2$, $F = \frac{1}{2}(\tau C + \sqrt{4C+\tau^2C^2})$, $R = (I+\tau F)^{-1}$. Recall that A is a self-adjoint positive definite operator, then the operator F will be a self-adjoint positive definite operator, too (see [14]). In addition, the bounded operator F is defined on the whole space H. Let $[\cdot]$ be the greatest integer function. Denote by $l = \left[\frac{\lambda}{\tau}\right]$. By using approximate formulas

$$\begin{aligned}&\tau u_t(0) = u(\tau) - u(0) - \tfrac{\tau^2}{3}u_{tt}(0) - \tfrac{\tau^2}{6}u_{tt}(\tau) + o(\tau^4),\\ &\tau u_t(1) = u(1) - u(1-\tau) + \tfrac{\tau^2}{3}u_{tt}(1) + \tfrac{\tau^2}{6}u_{tt}(1-\tau) + o(\tau^4),\\ &\tau u_t(\lambda) = u(t_{l+1}) - u(t_l) - \tfrac{\tau^2}{3}u_{tt}(t_l) - \tfrac{\tau^2}{6}u_{tt}(t_{l+1}) + o(\tau^4),\end{aligned} \tag{2.1}$$

and high order approximation of abstract elliptic equation ([11]- [13]), we can obtain a third order of ADS for problem (1.1)

$$\begin{cases} -\tau^{-2}(u_{k+1}-2u_k+u_{k-1}) + Cu_k = pt_k + f(t_k) \\ +\frac{\tau^2}{12}\left(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2} + Af(t_k)\right), t_k = k\tau,\ 1\le k\le N-1,\ N\tau=1, \\ \left(I-\frac{\tau^2}{6}A\right)u_1 - \left(I+\frac{\tau^2}{3}A\right)u_0 = \varphi\tau + \frac{\tau^2}{3}f_0 + \frac{\tau^2}{6}f_1 + \frac{\tau^3}{6}p, \\ \left(I+\frac{\tau^2}{3}A\right)u_N - \left(I-\frac{\tau^2}{6}A\right)u_{N-1} = \psi\tau - \frac{\tau^2}{3}f_N - \frac{\tau^2}{6}f_{N-1} - (1+\tau)\frac{\tau^2}{6}p, \\ \left(I-\frac{\tau^2}{6}A\right)u_{l+1} - \left(I+\frac{\tau^2}{3}A\right)u_l = \xi\tau + \frac{\tau^2}{3}f_l + \frac{\tau^2}{6}f_{l+1} + \left(t_l+\frac{\tau}{3}\right)\frac{\tau^2}{2}p. \end{cases} \tag{2.2}$$

For solving difference problem (2.2), we reduce it to some auxiliary difference problem. Namely, for a solution $\{u_k\}_{k=1}^{N-1}$ of difference problem (2.2), we apply the substitution

$$u_k = v_k + A^{-1}(pt_k) \tag{2.3}$$

and get a nonlocal boundary value difference problem for obtaining $\{v_k\}_{k=0}^{N}$. Later, putting $k = l$, we find $v_t(t_l)$. Then, by using formula

$$p = A\left(\xi - v_t(t_l)\right), \tag{2.4}$$

we define the element p. Finally, applying (2.3), we can obtain the solution $\{u_k\}_{k=1}^{N-1}$ of difference problem (2.2). According to this algorithm, we get the auxiliary difference problem for obtaining $\{v_k\}_{k=0}^N$:

$$\begin{cases} -\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Cv_k=f(t_k) \\ +\frac{\tau^2}{12}\left(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2}+Af(t_k)\right),\ t_k=k\tau,\ 1\le k\le N-1,\ N\tau=1, \\ \left(I-\frac{\tau^2}{6}A\right)v_1-\left(I+\frac{\tau^2}{3}A\right)v_0-\left(I-\frac{\tau^2}{6}A\right)v_{l+1} \\ +\left(I+\frac{\tau^2}{3}A\right)v_l=(\varphi-\xi)\tau+\frac{\tau^2}{3}(f_l-f_0)+\frac{\tau^2}{6}(f_{l+1}-f_1)\ , \\ \left(I+\frac{\tau^2}{3}A\right)v_N-\left(I-\frac{\tau^2}{6}A\right)v_{N-1}-\left(I-\frac{\tau^2}{6}A\right)v_{l+1} \\ +\left(I+\frac{\tau^2}{3}A\right)v_l=(\psi-\xi)\tau+\frac{\tau^2}{3}(f_N+f_l)+\frac{\tau^2}{6}(f_{N-1}+f_{l+1}). \end{cases} \tag{2.5}$$

Now, we give some lemmas.

Lemma 2.1 *The following estimates hold [13]:*

$$\left\|(I-R^{2N})^{-1}\right\|_{H\to H}\le M(\delta), k\tau\left\|FR^k\right\|_{H\to H}\le M(\delta),$$
$$\left\|R^k\right\|_{H\to H}\le M(\delta)(1+\delta\tau)^{-k}, k\ge 1,\delta>0,$$
$$\left\|F^{\beta}(R^{k+r}-R^k)\right\|_{H\to H}\le M(\delta)\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\le k<k+r\le N, 0\le\alpha,\beta\le 1.$$

Lemma 2.2 *The following estimate [13]*

$$\sum_{j=1}^{N-1}\tau\left\|FR^j\right\|_{H\to H}\le M(\delta)Y(\tau,\delta)$$

is valid, where

$$Y(\tau,\delta)=\min\left\{\ln\frac{1}{\tau},1+\left|\ln\|F\|_{H\to H}\right|\right\}.$$

Lemma 2.3 *The operators*

$$S_1=(I-R^{2N})^{-1}\left(R-R^{2N-1}+R^{N-1}-R^{N+1}-I+R^{2N}\right)$$

and

$$\begin{aligned} S_2 &= (I-R^{2N})^{-1}\left(-I+R^{2N}+R^l-R^{2N-l}+R-R^{2N-1}-R^{l+1}\right. \\ &\quad \left.+R^{2N-l-1}-R^{N-1}+R^{N+1}+R^{N-l-1}-R^{N+l+1}-R^{N-l}+R^{N+l}\right) \end{aligned}$$

have the inverses such that $G_1=S_1^{-1}, G_2=S_2^{-1}$, and the estimates

$$\| G_1\|_{H\to H}\le M(\delta), \| G_2\|_{H\to H}\le M(\delta) \tag{2.6}$$

are valid.

The proof of Lemma 2.3 is based on Lemma 2.1 and the following equalities

$$\begin{aligned} S_1 &= -(I-R)\left(I-R^{N-1}\right)\left(I+R^N\right)^{-1}, \\ S_2 &= -(I-R^N)^{-1}(I-R)\left(I-R^l\right)\left(I-R^{N-l-1}\right). \end{aligned}$$

Lemma 2.4 *The operator*

$$\begin{aligned} S_3^\tau &= S_2-\frac{\tau^2}{6}(I-R^{2N})^{-1}A\ \left[R-R^{2N-1}-R^{l+1}+R^{2N-l-1}+2I-2R^{2N}-2R^l\right.\\ &\left.+2R^{2N-l}-R^{N-1}+R^{N+1}+R^{N-l-1}-R^{N+l+1}+2R^{N-l}-2R^{N+l}\right] \end{aligned}$$

has an inverse $G_3^\tau=\left(S_3^\tau\right)^{-1}$, *and the inequality*

$$\| G_3^\tau \|_{H\to H}\le M(\delta) \tag{2.7}$$

is satisfied.

Proof. We can write

$$G_3^\tau-G=G_3^\tau G K^\tau, \tag{2.8}$$

where

$$\begin{aligned} &K^\tau=\tfrac{\tau^2}{6}(I-R^{2N})^{-1}A\ \left[R-R^{2N-1}-R^{l+1}+R^{2N-l-1}+2I-2R^{2N}-2R^l\right.\\ &\left.+2R^{2N-l}-R^{N-1}+R^{N+1}+R^{N-l-1}-R^{N+l+1}+2R^{N-l}-2R^{N+l}\right]. \end{aligned}$$

According to estimates of Lemma 2.1, we get

$$\begin{aligned} &\| K^\tau \|_{H\to H}=\| \tfrac{\tau^2}{6}(I-R^{2N})^{-1}A\\ &\times\left[R-R^{2N-1}-R^{l+1}+R^{2N-l-1}+2I-2R^{2N}-2R^l+2R^{2N-l}\right.\\ &\left.-R^{N-1}+R^{N+1}+R^{N-l-1}-R^{N+l+1}+2R^{N-l}-2R^{N+l}\right] \|_{H\to H}\le M_1(\delta)\tau. \end{aligned} \tag{2.9}$$

By using formula (2.8), estimates (2.6), (2.9), the triangle inequality, and Lemma 2.3 , we obtain

$$\begin{aligned} \| \quad & G_3^\tau \|_{H\to H}\le\| G \|_{H\to H}+\| G_3^\tau \|_{H\to H}\| G \|_{H\to H}\| K^\tau \|_{H\to H}\\ \le\quad & M(\delta)+\| G_3^\tau \|_{H\to H}\ M(\delta)M_1(\delta)\tau \end{aligned}$$

for any small parameter $\tau>0$. Hence, the estimate (2.7) is valid ■

Theorem 2.5 *Assume that* $\varphi,\xi,\psi\in D(A)$ *and* $\{f_k\}_{k=1}^{N-1}\in C_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. *Then, for any* $\{f_k\}_{k=1}^{N-1}$, φ,ψ,ξ *the difference problem (2.5) is uniquely solvable in* $C_\tau(H)$ *and solution of (2.5) satisfies the following stability and almost coercive stability estimates*

$$\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\le M\left[\|\varphi\|_H+\|\xi\|_H+\|\psi\|_H+\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{2.10}$$

$$\begin{aligned} &\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\left\|\{Cv_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\\ &\le M\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|F\|_{H\to H}|\right\}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\|F\varphi\|_H+\|F\xi\|_H+\|F\psi\|_H\right]. \end{aligned} \tag{2.11}$$

Here, M *is independent of* $\alpha,\tau,\ \varphi,\xi,\psi,$ *and* $\{f_k\}_{k=1}^{N-1}$.

Proof. Following [13], we get that the direct difference problem

$$\begin{cases} -\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Cv_k=\theta_k, \ 1\le k\le N-1, \\ v_0 \text{ and } v_N \text{ are given} \end{cases} \tag{2.12}$$

has a solution

$$\begin{aligned} v_k \ = \ & (I-R^{2N})^{-1}[\left(\left(R^k-R^{2N-k}\right)v_0+\left(R^{N-k}-R^{N+k}\right)v_N\right) \\ & -\left(R^{N-k}-R^{N+k}\right)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\theta_i\tau] \\ & +(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|-1}-R^{k+i-1}\right)\theta_i\tau, \ k=1,\dots,N-1, \end{aligned} \tag{2.13}$$

and the following estimates

$$\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\le M\left[\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\|Rv_0\|_{C_\tau(H)}+\|Rv_N\|_{C_\tau(H)}\right], \tag{2.14}$$

$$\begin{aligned} & \left\|\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\left\|\{Cv_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \\ \le M & \left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|F\|_{H\to H}|\right\}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\|CRv_0\|_H+\|CRv_N\|_H\right] \end{aligned} \tag{2.15}$$

are valid. By using (2.13) and nonlocal conditions, we get

$$\begin{aligned} v_0 = & -G_3^\tau(I-R^{2N})^{-1}\left[\left(I-\frac{\tau^2}{6}A\right)\left(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}\right)\right. \\ & \left.+\left(I+\frac{\tau^2}{3}A\right)\left(R^{N-l}-R^{N+l}\right)\right] \\ & \times\left\{G_1(I-R^{2N})^{-1}\left[\left(I-\frac{\tau^2}{6}A\right)\left(R^{N-1}-R^{N+1}+R-R^{2N-1}\right)\right.\right. \\ & \left.+\left(I+\frac{\tau^2}{3}A\right)\left(-I+R^{2N}\right)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\theta_i\tau\right] \\ & -G_1(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left[\left(I-\frac{\tau^2}{6}A\right)\left(R^{|1-i|-1}-R^i+R^{|N-1-i|-1}-R^{N+i-2}\right)\right. \\ & \left.+\left(I+\frac{\tau^2}{3}A\right)\left(R^{|l-i|-1}-R^{l+i-1}-R^{|N-i|-1}+R^{N+i-1}\right)\right]\theta_i\tau+G_1(\varphi-\psi)\tau \\ & \left.+\frac{\tau^2}{3}G_1(f_l-f_0-f_N-f_l)+\frac{\tau^2}{6}G_1(f_{l+1}-f_1-f_{N-1}-f_{l+1})\right\} \\ & +G_3^\tau(I-R^{2N})^{-1}\left[\left(I-\frac{\tau^2}{6}A\right)\left(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}\right)\right. \end{aligned} \tag{2.16}$$

$$+\left(I+\frac{\tau^2}{3}A\right)\left(R^{N-l}-R^{N+l}\right)\Big](2I+\tau F)^{-1}F^{-1}$$

$$\times\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\theta_i\tau-F_3^\tau(2I+\tau F)^{-1}F^{-1}$$

$$\times\sum_{i=1}^{N-1}\left[\left(I-\frac{\tau^2}{6}A\right)\left(R^{|1-i|-1}-R^i-R^{|l+1-i|-1}-R^{l+i}\right)\right.$$

$$\left.+\left(I+\frac{\tau^2}{3}A\right)\left(R^{|l-i|-1}-R^{l+i-1}\right)\right]\ \theta_i\tau$$

$$+G_3^\tau\left(\varphi-\xi\right)\tau+\frac{\tau^2}{3}G_3^\tau\left(f_l-f_0\right)+\frac{\tau^2}{6}G_3^\tau\left(f_{l+1}-f_1\right)$$

and

$$v_N=-v_0+G_1(I-R^{2N})^{-1}\left[\left(I-\frac{\tau^2}{6}A\right)\left(R^{N-1}-R^{N+1}+R-R^{2N-1}\right)\right.$$

$$\left.+\left(I+\frac{\tau^2}{3}A\right)\left(-I+R^{2N}\right)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)\theta_i\tau\right]$$

$$-G_1(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left[\left(I-\frac{\tau^2}{6}A\right)\left(R^{|1-i|-1}-R^i+R^{|N-1-i|-1}-R^{N+i-2}\right)\right.$$

$$\left.+\left(I+\frac{\tau^2}{3}A\right)\left(R^{|l-i|-1}-R^{l+i-1}-R^{|N-i|-1}+R^{N+i-1}\right)\right]\theta_i\tau+G_1(\varphi-\psi)\tau$$

$$+\frac{\tau^2}{3}G_1\left(f_l-f_0-f_N-f_l\right)+\frac{\tau^2}{6}G_1\left(f_{l+1}-f_1-f_{N-1}-f_{l+1}\right). \tag{2.17}$$

Therefore, there exists a unique solution of difference scheme (2.5) and it can be found by formulas (2.13), (2.16) and (2.17). By using (2.13), (2.16), (2.17), and Lemmas 2.1-2.3, we can get that the solution of difference problem (2.5) obeys the following estimates

$$\|Rv_0\|_{C_\tau(H)}\le M\left[\|\varphi\|_H+\|\xi\|_H+\|\psi\|_H+\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{2.18}$$

$$\|Rv_N\|_{C_\tau(H)}\le M\left[\|\varphi\|_H+\|\xi\|_H+\|\psi\|_H+\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{2.19}$$

$$\|CRv_0\|_H\le M\left[\min\left\{\ln\tfrac{1}{\tau},1+|\ln\|F\|_{H\to H}|\right\}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\|F\varphi\|_H\right.$$
$$\left.+\|F\xi\|_H+\|F\psi\|_H\right], \tag{2.20}$$

$$\|CRv_N\|_H\le M\left[\min\left\{\ln\tfrac{1}{\tau},1+|\ln\|F\|_{H\to H}|\right\}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right.$$
$$\left.+\|F\varphi\|_H+\|F\xi\|_H+\|F\psi\|_H\right], \tag{2.21}$$

In [13], the estimates

$$\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \leq M\left[\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \|Rv_0\|_{C_\tau(H)} + \|Rv_N\|_{C_\tau(H)}\right], \quad (2.22)$$

$$\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \left\|\{Cv_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}$$

$$\leq M\left[\min\left\{\ln\frac{1}{\tau}, 1+|\ln\|F\|_{H\to H}|\right\}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \|CRv_0\|_H + \|CRv_N\|_H\right] \quad (2.23)$$

are established for the solution of difference problem (2.12). So, applying (2.18), (2.19), and (2.22), we get estimate (2.10). Estimate (2.11) follows from inequalities (2.20), (2.21), and (2.23). ■

Let F be a self-adjoint positive definite operator in a Hilbert space H and $0 < \alpha < 1$. Denote by $E_\alpha = E_\alpha(D(F), H)$, the Banach space of those functions $q \in H$ for which the norm $\|q\|_{E_\alpha} = \sup_{z>0} z^{1-\alpha}\left\|Be^{-zB}q\right\|_H + \|q\|_H$ is finite.

Theorem 2.6 *Suppose that $\varphi, \xi, \psi \in D(F)$ and $\{f_k\}_{k=1}^{N-1} \in C_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, the solution $\{v_k\}_{k=1}^{N-1}$ of difference scheme (2.5) obeys the following coercive inequality*

$$\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \left\|\{Cv_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}$$

$$\leq M\left[\frac{1}{\alpha(1-\alpha)}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \|F\varphi\|_{E_\alpha} + \|F\psi\|_{E_\alpha} + \|F\xi\|_{E_\alpha}\right], \quad (2.24)$$

where M does not depend on $\alpha, \tau, \varphi, \xi, \psi$, and $\{f_k\}_{k=1}^{N-1}$.

Proof. Applying formulas (2.5), (2.16),(2.17), Lemmas 2.1-2.3, and definitions of norm of spaces E_α and $C_\tau^\alpha(H)$, we have the following estimates

$$\|CRv_0-\theta_1\|_{E_\alpha} \leq M\left[\tfrac{1}{\alpha(1-\alpha)}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \|F\varphi\|_{E_\alpha} + \|F\psi\|_{E_\alpha} + \|F\xi\|_{E_\alpha}\right], \quad (2.25)$$

$$\|CRv_N-\theta_{N-1}\|_{E_\alpha} \leq M\left[\tfrac{1}{\alpha(1-\alpha)}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)}\cdot + \|F\varphi\|_{E_\alpha} + \|F\psi\|_{E_\alpha} + \|F\xi\|_{E_\alpha}\right], \quad (2.26)$$

In [13], for the solution of difference problem (2.12) estimate

$$\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \left\|\{Cv_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}$$

$$\leq M\left[\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \|CRv_0-\theta_1\|_{E_\alpha} + \|CRv_N-\theta_{N-1}\|_{E_\alpha}\right] \quad (2.27)$$

is proved. Therefore, from estimates (2.26) and (2.27) yield estimate (2.24). ■

From Theorems 2.5 and 2.6, formulas (2.3) and (2.4), and the triangle inequality we can establish the following theorems on stability estimates for the solution $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ of difference problem (2.2).

Theorem 2.7 *Assume that* $\varphi, \xi, \psi \in D(A)$ *and* $\{f_k\}_{k=1}^{N-1} \in \mathcal{C}_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. *Then, the solution* $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ *of difference problem (2.2) in* $C_\tau(H)\times H$ *obeys the following stability estimates:*

$$\left\|\{u_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \le M\left[\|\varphi\|_H + \|\xi\|_H + \|\psi\|_H + \left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right],$$

$$\left\|A^{-1}p\right\|_H \le M\left[\|\varphi\|_H + \|\xi\|_H + \|\psi\|_H + \left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right],$$

$$\|p\|_H \le M\left[\|A\varphi\|_H + \|A\xi\|_H + \|A\psi\|_H + \frac{1}{\alpha(1-\alpha)}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}\right],$$

where M *is independent of* $\alpha, \tau, \varphi, \xi, \psi$, *and* $\{f_k\}_{k=1}^{N-1}$.

Theorem 2.8 *Suppose that* $\varphi, \xi, \psi \in D(A)$ *and* $\{f_k\}_{k=1}^{N-1} \in \mathcal{C}_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. *Then, for the solution* $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ *of difference scheme (2.2) in* $C_\tau(H)\times H$ *obeys almost coercive stability estimate*

$$\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \left\|\{Cu_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \|p\|_H$$

$$\le M\left[\min\left\{\ln\frac{1}{\tau}, 1+\left|\ln\|F\|_{H\to H}\right|\right\}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \|F\varphi\|_H + \|F\xi\|_H + \|F\psi\|_H\right],$$

where M *does not depend on* $\alpha, \tau, \varphi, \xi, \psi$, *and* $\{f_k\}_{k=1}^{N-1}$.

Theorem 2.9 *Assume that* $\varphi, \xi, \psi \in D(A)$ *and* $\{f_k\}_{k=1}^{N-1} \in \mathcal{C}_\tau^{\alpha}(H)\,(0<\alpha<1)$. *Then, the solution* $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ *of difference problems (2.2) obeys the following coercive inequality*

$$\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha}(H)} + \left\|\{Cu_k\}_{k=1}^{N-1}\right\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)} + \|p\|_H$$

$$\le M\left[\frac{1}{\alpha(1-\alpha)}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \|F\varphi\|_{E_\alpha} + \|F\psi\|_{E_\alpha} + \|F\xi\|_{E_\alpha}\right],$$

where M *is independent of* $\alpha, \tau, \varphi, \xi, \psi$, *and* $\{f_k\}_{k=1}^{N-1}$.

3 Difference scheme for multi-dimensional problem

Recall that differential expression $A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} + \sigma u$ defines a self-adjoint strongly positive definite operator A^x acting on $L_2(\overline{\Omega})$ with the domain [14]

$$D(A^x) = \left\{ u(x) \in W_2(\overline{\Omega}),\ u(x) = 0, \quad x \in S \right\}.$$

The approximation of problem (1.2) is carried out in two steps. In the first step, define the grid spaces

$$\begin{aligned} \widetilde{\Omega}_h &= \{x = (h_1 m_1, \ldots, h_n m_n); m = (m_1, \ldots, m_n), \\ m_r &= 0, \ldots, M_r,\ h_r M_r = 1, r = 1, \ldots, n\}, \Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S, \end{aligned}$$

where $M_1, ..., M_n$ are given natural numbers. We assign the difference operator A_h^x by formula

$$A_h^x u^h(x) = -\sum_{r=1}^{n} \left(a_r(x) u_{\overline{x}_r}^h(x) \right)_{x_r, j_r} + \sigma u^h(x) \tag{3.1}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$.

Let $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ be spaces of the grid functions $q^h(x) = \{q(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\|q\|_{L_{2h}} = \left(\sum_{x \in \widetilde{\Omega}_h} |q^h(x)|^2 h_1 \cdots h_n \right)^{1/2}, \left\|q^h\right\|_{W_{2h}^2} = \left\|q^h\right\|_{L_{2h}}$$

$$+ \left(\sum_{x \in \widetilde{\Omega}_h} \sum_{r=1}^{n} \left|(q^h)_{x_r}\right|^2 h_1 \cdots h_n \right)^{1/2} + \left(\sum_{x \in \widetilde{\Omega}_h} \sum_{r=1}^{n} \left|(q^h(x))_{x_r \overline{x_r},\ m_r}\right|^2 h_1 \cdots h_n \right)^{1/2}.$$

For $\left\{u^h(t,x), p^h(x)\right\}$, we get a system of ordinary differential equations

$$\begin{cases} -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x) + t p^h(x),\ 0 < t < 1,\ x \in \widetilde{\Omega}_h, \\ u_t^h(0,x) = \varphi(x),\ u_t^h(\lambda,x) = \xi(x), \quad u_t^h(T,x) = \psi(x), x \in \widetilde{\Omega}_h. \end{cases} \tag{3.2}$$

In the second step of approximation, applying (2.2), system equations (3.2) is replaced by a third order of ADS

$$\begin{cases} -\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h) + A_h^x u_k^h + \frac{\tau^2}{12}\left(A_h^x\right)^2 u_k^h = p^h t_k + f^h(t_k) \\ +\frac{\tau^2}{12}\left(\frac{f^h(t_{k+1})-2f^h(t_k)+f(t_{k-1})}{\tau^2} + Af^h(t_k)\right), t_k = k\tau, \\ 1 \le k \le N-1, N\tau = 1, \\ \left(I - \frac{\tau^2}{6}A_h^x\right)u_1^h - \left(I + \frac{\tau^2}{3}A_h^x\right)u_0^h = \tau\varphi^h + \frac{\tau^2}{3}f_0^h + \frac{\tau^2}{6}f_1^h + \frac{\tau^3}{6}p^h, \\ \left(I - \frac{\tau^2}{6}A_h^x\right)u_{l+1}^h - \left(I + \frac{\tau^2}{3}A_h^x\right)u_l^h = \tau\xi^h + \frac{\tau^2}{3}f_l^h + \frac{\tau^2}{6}f_{l+1}^h + \left(t_l + \frac{\tau}{3}\right)\frac{\tau^2}{2}p^h, \\ -\left(I - \frac{\tau^2}{6}A_h^x\right)u_{N-1}^h + \left(I + \frac{\tau^2}{3}A_h^x\right)u_N^h \\ = \tau\psi^h - \frac{\tau^2}{3}f_N^h - \frac{\tau^2}{6}f_{N-1}^h - (1+\tau)\frac{\tau^2}{6}p^h. \end{cases} \quad (3.3)$$

Applying (2.3) , we have auxiliary difference problem

$$\begin{cases} -\frac{v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)}{\tau^2} + A_h^x v_k^h(x) + \frac{\tau^2}{12}\left(A_h^x\right)^2 v_k^h(x) = f^h(t_k,x) \\ +\frac{\tau^2}{12}\left(\frac{f^h(t_{k+1},x)-2f^h(t_k,x)+f^h(t_{k-1},x)}{\tau^2} + A_h^x f^h(t_k,x)\right), \\ \left(I - \frac{\tau^2}{6}A_h^x\right)v_1^h(x) - \left(I + \frac{\tau^2}{3}A_h^x\right)v_0^h(x) - \left(I - \frac{\tau^2}{6}A_h^x\right)v_{l+1}^h(x) \\ +\left(I + \frac{\tau^2}{3}A_h^x\right)v_l^h(x) = \left(\varphi^h(x) - \xi^h(x)\right)\tau + \frac{\tau^2}{3}\left[f_0^h(x) - f_l^h(x)\right] \\ +\frac{\tau^2}{6}\left[f_1^h(x) - f_{l+1}^h(x)\right],\ x \in \widetilde{\Omega}_h \\ \left(I + \frac{\tau^2}{3}A_h^x\right)v_N^h(x) - \left(I - \frac{\tau^2}{6}A_h^x\right)v_{N-1}^h(x) - \left(I - \frac{\tau^2}{6}A_h^x\right)v_{l+1}^h(x) \\ +\left(I + \frac{\tau^2}{3}A_h^x\right)v_l^h(x) = \left(\psi^h(x) - \xi^h(x)\right)\tau - \frac{\tau^2}{3}\left[f_N^h(x) + f_l^h(x)\right] \\ -\frac{\tau^2}{6}\left[f_{N-1}^h(x) + f_{l+1}^h\ (x)\right],\ x \in \widetilde{\Omega}_h, l = \left[\frac{\lambda}{\tau}\right]. \end{cases} \quad (3.4)$$

We define function $p^h(x)$ by formula

$$p^h(x) = A_h^x\left(\xi^h(x) - v_t^h(t_l,x)\right), x \in \widetilde{\Omega}_h. \quad (3.5)$$

Let $\tau > 0$ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2} > 0$ be sufficiently small numbers.

Theorem 3.1 *For solution of difference scheme (3.3) we have the following stability estimates:*

$$\begin{aligned} \left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})} &\le M\ [\left\|\varphi^h\right\|_{L_{2h}} + \left\|\xi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})}], \\ \left\|\left(A^x\right)^{-1}p^h\right\|_{L_{2h}} &\le M\ \left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\xi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C_\tau(H)}\right], \end{aligned}$$

where M does not depend on $\alpha, \tau, h, \varphi^h(x), \xi^h(x),\ \psi^h(x),$ and $\left\{f_k^h(x)\right\}_1^{N-1}$.

Theorem 3.2 *The solution of difference scheme (3.3) obeys the following almost coercive* *stability estimate:*

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^k}{\tau^2}\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|\ (u_k^h)_{x_r\overline{x_r},\ m_r}\right\|_{L_{2h}}+\left\|p^h\right\|_{L_{2h}}$$
$$\le\ M\ \left[\ln\left(\frac{1}{\tau+h}\right)\left\|\left\{f_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})}+\left\|\varphi^h\right\|_{W_{2h}^1}+\left\|\xi^h\right\|_{W_{2h}^1}+\left\|\psi^h\right\|_{W_{2h}^1}\right],$$

where M is independent of $\alpha,\tau,h,\varphi^h(x),\xi^h(x),\ \psi^h(x),$ and $\left\{f_k^h(x)\right\}_1^{N-1}$.

The proofs of Theorems 3.1 and 3.2 are based on symmetry properties of operator A_h^x in L_{2h}, the results of abstract Theorems 2.7 and 2.8, and the following theorem on the coercivity inequality.

Theorem 3.3 [15] *For the solution of the elliptic difference problem*

$$\begin{cases} A_h^x u^h(x)=\omega^h(x),\ x\in\widetilde{\Omega}_h,\\ u^h(x)=0,\ x\in S_h,\end{cases}$$

the following coercivity inequality holds :

$$\sum_{r=1}^n\left\|(u_k^h)_{\overline{x}_r x_r,j_r}\right\|_{L_{2h}}\le M||\omega^h||_{L_{2h}},$$

where M does not depend on ω^h and h.

4 Numerical example

In this section, by using a third order of ADS, we give errors of numerical calculations for the inverse elliptic problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial^2 u(t,x)}{\partial x^2}+u(t,x)=\exp(-\pi t)\sin(\pi x)+tp(x),\\ 0<x<1,0<t<1,\\ u_t(0,x)=(-\pi+1)\sin(x),0\le x\le 1,\\ u(1,x)=(-\pi\exp(-\pi)+1)\sin(x),0\le x\le 1,\\ u(\lambda,x)=(-\pi\exp(-\pi\lambda)+1)\sin(x),\ 0\le x\le 1,\\ u(t,0)=u(t,\pi)=0,\ 0\le t\le 1\quad(\lambda=\frac{1}{2}).\end{cases}\tag{4.1}$$

It is clear that $\left\{(\exp(-\pi t)+t)\sin(\pi x),\left(\pi^2+1\right)\sin(\pi x)\right\}$ is the exact solution of (4.1). Let us take $\tau=\frac{1}{N}$ and $h=\frac{1}{M}$. For approximate solution of inverse problem (4.1), introduce the set of grid points

$$(t_k,x_i):\{(t_k,x_i)|t_k=k\tau,k=1,\ldots,N-1,x_i=ih,\ i=1,\ldots,M-1\}.$$

By MATLAB program, we record numerical solutions for different values of N, M. u_n^k and v_n^k are the numerical solutions of difference schemes at grid points of (t_k, x_n), respectively, and p_n is the numerical solutions at x_n. The errors of calculations are computed by

$$Ev_M^N = \max_{1\le k\le N-1} \Big(\sum_{n=1}^{M-1} \left|v(t_k,x_n)-v_n^k\right|^2 h\Big)^{\frac{1}{2}},$$
$$Eu_M^N = \max_{1\le k\le N-1} \Big(\sum_{n=1}^{M-1} \left|u(t_k,x_n)-u_n^k\right|^2 h\Big)^{\frac{1}{2}},\ Ep_M = \Big(\sum_{n=1}^{M-1} \left|p(x_n)-p_n\right|^2 h\Big)^{\frac{1}{2}},$$

respectively.

Tables 1-3 are constructed for $N = 10, M = 300, N = 20, M = 1200$. Hence, third order ADS is more accurate comparing with a first and a second order of ADS. Table 1 presents the error between the exact solution and solutions derived by difference schemes for nonlocal problem. Table 2 include error between the exact and approximate of p function. Table 3 gives the error for u function. As it can be seen from Tables 1-3, the third order of ADS is more accurate comparing with a first and a second order of ADS.

Table 1. Error Ev_M^N

Difference Schemes for v	N=10,M=300	N=20,M=1200
First order of ADS	0.1096	0.052929
Second order of ADS	0.0217	5.84×10^{-3}
Third order of ADS	7.06×10^{-4}	9.61×10^{-5}

Table 2. Error Ep_M

Calculation of p	N=10,M=300	N=20,M=1200
First order of ADS	0.21335	0.095715
Second order of ADS	0.25584	0.065631
Third order of ADS	0.03099	4.60×10^{-3}

Table 3. Error Eu_M^N

Difference Schemes for u	N=10,M=300	N=20,M=1200
First order of ADS	0.1096	0.052929
Second order of ADS	0.0234	6.13×10^{-3}
Third order of ADS	2.98×10^{-3}	4.40×10^{-4}

5 Conclusion

In this paper, a third order of ADS for approximate solution of Neumann type overdetermination problem for elliptic differential equation with Dirichlet boundary condition is presented. Theorems on the stability, almost coercive stability and coercive stability estimates for its solution are proved. Furthermore, we study a third order of ADS for multi-dimensional elliptic inverse problem. The results of numerical experiments are given. Computer calculations show that a third order of ADS is more accurate comparing to a first and a second order of ADS [8].

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