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Numerical solution to Bitsadze-Samarskii type elliptic overdetermined multipoint NBVP

Charyyar Ashyralyev* 

*Correspondence:
charyyar@gumushane.edu.tr
Department of Mathematical
Engineering, Gumushane University,
Gumushane, 29100, Turkey
TAU, Ashgabat, Turkmenistan

Abstract

This paper is devoted to a Bitsadze-Samarskii type overdetermined multipoint nonlocal boundary value problem (NBVP). This inverse problem is reduced to an auxiliary multipoint NBVP. Stability estimates for the solution of the auxiliary NBVP are established. The finite difference method is applied to get the first and second order of accuracy of approximate solutions of the abstract overdetermined problem. Stability estimates for the solution of difference problems are proved. Then the established abstract results are applied to get stability estimates for the solution of the Bitsadze-Samarskii type overdetermined elliptic multidimensional differential and difference problems with multipoint nonlocal boundary conditions (NBVC). Finally, numerical results with explanation on the realization for two dimensional and three dimensional elliptic overdetermined multipoint NBVPs in test examples are presented.

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1 Introduction

The theory and methods of the solutions of inverse problems of determining the parameter of partial differential equations have been extensively studied by several researchers (see [1–19] and the references therein).

The well-posedness of source identification problems for elliptic type differential and difference equations was studied in [12–19]. These papers were devoted to various inverse elliptic problems and their approximations. For abstract elliptic equations in general L_p spaces and some exposition on practical applications of inverse problems we refer to [20, 21] and the references therein.

In recent years, Bitsadze-Samarskii type NBVPs have been investigated extensively (see [22–27]).

Let A be a self-adjoint positive definite operator (SAPD) in an arbitrary Hilbert space H . Then $B = A^{\frac{1}{2}}$ is also an SAPD operator. One assumes that the smooth function $f(\cdot)$, elements $\varphi, \psi, \zeta \in D(B)$, numbers $\lambda_0, \lambda_i, i = 1, \dots, q$, and nonnegative real coefficients $k_i, i = 1, \dots, q$, are given. In [8], the Bitsadze-Samarskii type inverse elliptic problem of finding

an element $p \in H$ and a function $u(\cdot) \in C^2([0, T], H) \cap C([0, T], D(A))$ such that

$$\begin{cases} -u_{tt}(t) + Au(t) = f(t) + p, & 0 < t < T, \\ u_t(0) = \varphi, \quad u_t(T) = \sum_{i=1}^q k_i u_t(\lambda_i) + \psi, \quad u(\lambda_0) = \zeta, \end{cases} \tag{1}$$

was studied. Here $\lambda_1, \dots, \lambda_q$ such that $0 < \lambda_1 < \lambda_2 < \dots < \lambda_q < T$.

Under the conditions

$$\sum_{i=1}^q k_i \leq 1 \tag{2}$$

the theorem on the solvability and uniqueness of the solution of problem (1) was proved.

Denote by $C(H)$, $C^\alpha(H)$, and $C_{0T}^{\alpha,\alpha}(H)$ the corresponding Banach spaces of smooth H -valued functions u on $[0, T]$ with the following norms:

$$\begin{aligned} \|u\|_{C(H)} &= \max_{0 \leq t \leq T} \|u(t)\|_H, \\ \|u\|_{C^\alpha(H)} &= \|u\|_{C(H)} + \sup_{0 \leq t < t+s \leq T} s^{-\alpha} \|u(t+s) - u(t)\|_H, \\ \|u\|_{C_{0T}^{\alpha,\alpha}(H)} &= \|u\|_{C(H)} + \sup_{0 \leq t < t+s \leq T} \left[\frac{(t+s)(T-t)}{s} \right]^\alpha \|u(t+s) - u(t)\|_H. \end{aligned}$$

Lemma 1 *The following inequalities are fulfilled ([28]):*

$$\begin{aligned} \|Be^{-tB}\|_{H \rightarrow H} &\leq t^{-\alpha}, \quad 0 \leq \alpha \leq 1, t > 0, \quad \|(I - e^{-2TB})^{-1}\|_{H \rightarrow H} \leq M(\delta), \\ \|B^\beta(e^{-tB} - e^{-(t+\tau)B})\|_{H \rightarrow H} &\leq M(\delta) \frac{\tau^\alpha}{(t+\tau)^{\alpha+\beta}}, \quad 0 \leq \alpha, \beta \leq 1, \\ 0 < t \leq t + \tau \leq 1, \quad \|(I \pm e^{-\lambda B})^{-1}\|_{H \rightarrow H} &\leq M(\lambda), \quad 0 < \lambda \leq T. \end{aligned} \tag{3}$$

In beginning of this paper, we discuss stability estimates for the solution of the overdetermined problem (1). Next, we apply the finite difference method and establish stability estimates for the first and second order of accuracy difference schemes. Later, we consider Bitsadze-Samarskii type overdetermined elliptic multidimensional differential and difference problems with multipoint NBVC and we obtain stability estimates for their solutions. Finally, we present numerical results with an explanation on the realization for two dimensional and three dimensional elliptic overdetermined multipoint NBVP in test examples.

2 Overdetermined differential problem

Let us take $Q = (I - e^{-2TB})^{-1}$. It is well known that ([9]) the function

$$G(t, s) = -\frac{1}{2B} Q [e^{-(t+s)B} + e^{-(2T-t-s)B} + e^{-|t-s|B} + e^{-(2T-|t-s|)B}] \tag{4}$$

is the Green function for the following elliptic problem with second kind boundary conditions:

$$\begin{cases} -v_{tt}(t) + Av(t) = f(t), & 0 < t < T, \\ v_t(0) = \varphi, \quad v_t(T) = \xi. \end{cases} \tag{5}$$

Under the assumptions that $\varphi, \xi \in D(B), f(\cdot) \in C^1([0, T], H)$, the solution of problem (5) is defined by the equation ([9])

$$v(t) = -Q[(e^{-tB} + e^{-(2T-t)B})B^{-1}\varphi - (e^{-(T-t)B} + e^{-(T+t)B})B^{-1}\xi] + \int_0^T G(t,s)f(s) ds. \tag{6}$$

Moreover, the derivative is given by the formula

$$v_t(t) = Q[(e^{-tB} - e^{-(2T-t)B})\varphi + (e^{-(T-t)B} - e^{-(T+t)B})\xi] + F(B, f, t),$$

$$F(B, f, t) = \frac{1}{2}Q \int_0^T (e^{-(t+s)B} - e^{-(2T-t-s)B})f(s) ds + \frac{1}{2}Q \int_0^t (e^{-(t-s)B} - e^{-(2T-t+s)B})f(s) ds - \frac{1}{2}Q \int_t^T (e^{-(s-t)B} - e^{-(2T+t-s)B})f(s) ds. \tag{7}$$

Let us take $\alpha \in (0, 1)$. To formulate our results, we use the notation

$$P_H^\gamma(\varphi, \psi, \zeta) = \|A^\gamma \varphi\|_H + \|A^\gamma \psi\|_H + \|A^\gamma \zeta\|_H, \quad \gamma = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1.$$

Theorem 1 *Suppose that A is an SAPD operator, condition (2) for coefficients is valid, $\varphi, \psi, \zeta \in D(A) \cap D(A^{-\frac{1}{2}})$, and $f(t) \in C_{0T}^{\alpha, \alpha}(H)$. Then for the solution $(p, u(t))$ of problem (1) the following stability estimates hold:*

$$\|u\|_{C(H)} \leq M(\delta)[P_H^0(\varphi, \psi, \zeta) + P_H^{-\frac{1}{2}}(\varphi, \psi, \zeta) + \|f\|_{C(H)}], \tag{8}$$

$$\|A^{-1}p\|_H \leq M(\delta)[P_H^0(\varphi, \psi, \zeta) + \|f\|_{C(H)}], \tag{9}$$

$$\|p\|_H \leq M(\delta)\left[P_H^1(\varphi, \psi, \zeta) + \frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha, \alpha}(H)}\right], \tag{10}$$

where $M(\delta)$ does not depend on $\alpha, \varphi, \psi, \zeta$, and $f(t)$.

Proof Applying the substitution

$$u(t) = v(t) + A^{-1}(p), \tag{11}$$

we get the auxiliary nonlocal boundary value problem:

$$\begin{cases} -v_{tt}(t) + Av(t) = f(t), & 0 < t < T, \\ v_t(0) = \varphi, & v_t(T) = \sum_{i=1}^q k_i v_t(\lambda_i) + \psi, \end{cases} \tag{12}$$

to find a function $v(\cdot)$.

Thus, we consider the algorithm to solve (1) which includes three stages. In the first stage, we solve the nonlocal boundary value problem (12). In the second stage, we put

$t = \lambda_0$, and find $v(\lambda_0)$. Then we obtain an element p by the formula

$$p = A\zeta - Av(\lambda_0). \tag{13}$$

In the third stage, we use equation (11) to find $u(t)$.

In [22–27], the well-posedness of the Bitsadze-Samarskii type nonlocal BVP (12) in the case of $q = 1$ and its applications were investigated. The stability and coercive stability estimates for the solution of problem (12) were established.

Taking into account equation (7) in the nonlocal condition of problem (12), we get the equation

$$\begin{aligned} & \left[I - \sum_{i=1}^q k_i Q(e^{-(T-\lambda_i)B} - e^{-(T+\lambda_i)B}) \right] v_t(T) \\ &= \sum_{i=1}^q k_i [Q(e^{-\lambda_i B} - e^{-(2T-\lambda_i)B})\varphi + F(B, f, \lambda_i)] + \psi \end{aligned} \tag{14}$$

to find $v_t(T)$. In ([23]), the authors proved that the operator

$$G = I - \sum_{i=1}^q k_i Q(e^{-(T-\lambda_i)B} - e^{-(T+\lambda_i)B})$$

has an inverse and its norm is bounded by

$$\|G^{-1}\|_{H \rightarrow H} \leq M(\delta). \tag{15}$$

So, we can obtain a solution of problem (5) by

$$\begin{aligned} v(t) = & -Q(e^{-tB} + e^{-(2T-t)B})B^{-1}\varphi - Q(e^{-(T-t)B} + e^{-(T+t)B})G^{-1} \\ & \times \left\{ \sum_{i=1}^q k_i [Q(e^{-\lambda_i B} - e^{-(2T-\lambda_i)B})B^{-1}\varphi + B^{-1}F(B, f, \lambda_i)] \right. \\ & \left. + B^{-1}\psi \right\} + \int_0^T G(t, s)f(s) ds. \end{aligned} \tag{16}$$

Applying estimates (2), (3), (15), and the Cauchy-Schwartz and the triangle inequalities, we can obtain

$$\begin{aligned} & \|Q(e^{-tB} + e^{-(2T-t)B})B^{-1}\varphi\|_{C(H)} \leq M(\delta)\|B^{-1}\varphi\|_H, \\ & \left\| Q(e^{-(T-t)B} + e^{-(T+t)B})G^{-1} \left\{ \sum_{i=1}^q k_i B^{-1}\varphi + B^{-1}\psi \right\} \right\|_{C(H)} \leq M(\delta)\|B^{-1}\varphi\|_H, \\ & \left\| Q(e^{-(T-t)B} + e^{-(T+t)B})G^{-1} \sum_{i=1}^q k_i B^{-1}F(B, f, \lambda_i) \right\|_{C(H)} \leq M(\delta)\|f\|_{C(H)}, \\ & \left\| \int_0^T G(t, s)f(s) ds \right\|_{C(H)} \leq M(\delta)\|f\|_{C(H)}. \end{aligned} \tag{17}$$

From (17) it follows that

$$\|v\|_{C(H)} \leq M(\delta) [P_H^{-\frac{1}{2}}(\varphi, \psi, \zeta) + \|f\|_{C(H)}]. \tag{18}$$

By using (13),(16) and (3), the estimates (9) and (10) can be established.

From (11), (13), (16), (18), and the triangle inequalities the estimate (8) follows. □

Theorem 2 *Under the assumptions that A is an SAPD operator, condition (2) for the coefficients is valid, $\varphi, \psi, \zeta \in D(A) \cap D(A^{\frac{1}{2}})$ and $f(t) \in C_{0T}^{\alpha,\alpha}(H)$ ($0 < \alpha < 1$), and the solution $(p, u(t))$ of problem (1) obeys the following coercive inequality:*

$$\|u'\|_{C_{0T}^{\alpha,\alpha}(H)} + \|Au\|_{C_{01}^{\alpha,\alpha}(H)} + \|p\|_H \tag{19}$$

$$\leq M(\delta) \left[P_H^1(\varphi, \psi, \zeta) + P_H^{\frac{1}{2}}(\varphi, \psi, \zeta) + \frac{1}{\alpha(1-\alpha)} \|f\|_{C_{0T}^{\alpha,\alpha}(H)} \right], \tag{20}$$

where $M(\delta)$ is independent of $\alpha, \varphi, \psi, \zeta$, and $f(t)$.

Proof Applying (11) and (16), we have

$$\begin{aligned} Au(t) = & -Q(e^{-tB} + e^{-(2T-t)B} - e^{-\lambda_0 B} - e^{-(2T-\lambda_0)B})B\varphi \\ & - Q(e^{-(T-t)B} + e^{-(T+t)B} - e^{-(T-\lambda_0)B} - e^{-(T+\lambda_0)B})G^{-1} \\ & \times \left\{ \sum_{i=1}^q k_i [Q(e^{-\lambda_i B} - e^{-(2T-\lambda_i)B})B\varphi + BF(B, f, \lambda_i)] + B\psi \right\} - \frac{1}{2}QB \\ & \times \int_0^T [e^{-(t+s)B} + e^{-(2T-t-s)B} + e^{-|t-s|B} + e^{-(2T-|t-s|)B}]f(s) ds \\ & + \frac{1}{2}QB \int_0^T [e^{-(\lambda_0+s)B} + e^{-(2T-\lambda_0-s)B} + e^{-|\lambda_0-s|B} \\ & + e^{-(2T-|\lambda_0-s|)B}]f(s) ds. \end{aligned} \tag{21}$$

The proof of estimate (20) is based on equation (21) and the estimates (3). □

3 Difference schemes for problem (1)

Since A is an SAPD operator, the operator $C = \frac{1}{2}(\tau A + \sqrt{4A + \tau^2 A^2})$ will be an SAPD operator, too ([29]). Denote $R = (I + \tau C)^{-1}$, $P = (I - R^{2N})^{-1}$, $D = (I + \tau C)(2I + \tau C)^{-1}C^{-1}$. The bounded operator R is defined on the whole space H .

Lemma 2 ([28]) *The following estimates hold:*

$$\begin{aligned} \|R^k\|_{H \rightarrow H} & \leq M(\delta)(1 + \delta^{\frac{1}{2}}\tau)^{-k}, & \|CR^k\|_{H \rightarrow H} & \leq \frac{M(\delta)}{k\tau}, \quad k \geq 1, \\ \|P\|_{H \rightarrow H} & \leq M(\delta), \quad \delta > 0. \end{aligned}$$

Lemma 3 *Under the assumptions that (2) is valid, the operator*

$$\Delta_1 = (I - R^{2N}) \left[I - R^{2N-2} - \sum_{i=1}^q k_i (R^{N-l_i-1} - R^{N+l_i-1}) \right] \tag{22}$$

has inverse G_1^{-1} and the following estimate holds:

$$\|\Delta_1^{-1}\|_{H \rightarrow H} \leq M(\delta, \lambda_1, \dots, \lambda_q). \tag{23}$$

Lemma 4 Under the assumptions that (2) is valid, the operator

$$\begin{aligned} \Delta_2 = & \left[R - 3I + R^{2N-2}(I - 3R) \right] \left[3I - R - R^{2N-2}(I - 3R) \right. \\ & \left. - \sum_{i=1}^q k_i \left[R^{N-l_i-1}(3I - R + \mu_i(I - R)) - R^{2N+l_i-1}(I - 3R + \mu_i(I - R)) \right] \right] \\ & - (I + R)R^{N-2}(-R^2 + 4R - I) \left[(I + R)R^{N-2}(R^2 - 4R + I) \right. \\ & \left. - \sum_{i=1}^q k_i \left[R^{l_i-1}(I - 3R + \mu_i(I - R)) - R^{2N-l_i-1}(3I - R + \mu_i(I - R)) \right] \right] \end{aligned}$$

has inverse Δ_2^{-1} and the following estimate holds:

$$\|\Delta_2^{-1}\|_{H \rightarrow H} \leq M(\delta, \lambda_1, \dots, \lambda_q). \tag{24}$$

It is well known that [28]

$$\begin{aligned} v_m = & P \left[(R^m - R^{2N-m})v_0 + (R^{N-m} - R^{N+m})v_N \right] \\ & - P(R^{N-m} - R^{N+m})D \sum_{j=1}^{N-1} (R^{N-j} - R^{N+j})f_j\tau \\ & + D \sum_{j=1}^{N-1} (R^{|m-j|} - R^{m+j})f_j\tau \quad (1 \leq m \leq N - 1) \end{aligned} \tag{25}$$

is a solution of the direct problem

$$\begin{cases} -\tau^{-2}(v_{m+1} - 2v_m + v_{m-1}) + Av_m = f_m, & 1 \leq m \leq N - 1, \\ v_0 \text{ and } v_N \text{ are given.} \end{cases} \tag{26}$$

Let $[\cdot]$ be the notation of the greatest integer function, and

$$l_i = \left[\frac{\lambda_i}{\tau} \right], \quad \mu_i = \frac{\lambda_i}{\tau} - l_i, \quad 0 \leq i \leq q.$$

We consider the first and second order of accuracy difference schemes (ADSs) for the nonlocal problem (1). First, by using the approximate formulas

$$u_t(\lambda_i) = \frac{u_{l_i+1} - u_{l_i}}{\tau} + o(\tau), \quad u_{l_0} = \zeta + o(\tau), \tag{27}$$

problem (1) is reduced to the first order of the ADS

$$\begin{cases} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = f_k + p, & 1 \leq k \leq N - 1, \\ u_1 - u_0 = \tau \varphi, \\ u_N - u_{N-1} = \sum_{i=1}^q k_i(u_{i+1} - u_i) + \tau \psi, & u_{l_0} = \zeta. \end{cases} \tag{28}$$

Second, applying the approximate formulas

$$\begin{aligned} u_t(\lambda_i) &= \frac{3u_{i+1} - 4u_i + u_{i-1}}{2\tau} + \mu_i \frac{u_{i+1} - 2u_i + u_{i-1}}{\tau} + o(\tau^2), \\ u(\lambda_0) &= u_{l_0} + \mu_i(u_{l_0+1} - u_{l_0}) + o(\tau^2), \end{aligned} \tag{29}$$

problem (1) can be reduced to the second order of the ADS,

$$\begin{cases} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = f_k + p, & 1 \leq k \leq N - 1, \\ \frac{-3u_0 + 4u_1 - u_2}{2\tau} = \varphi, \\ \frac{3u_N - 4u_{N-1} + u_{N-2}}{2\tau} = \psi + \sum_{i=1}^q k_i \left[\frac{3u_{i+1} - 4u_i + u_{i-1}}{2\tau} + 2\mu_i \frac{u_{i+1} - 2u_i + u_{i-1}}{\tau} \right], \\ u_{l_0} + \mu_i(u_{l_0+1} - u_{l_0}) = \zeta. \end{cases} \tag{30}$$

Introduce the set of grid points $\{t_k = k\tau, 1 \leq k \leq N - 1, N\tau = T\}$ and the spaces $C_\tau(H)$, $C_\tau^\alpha(H)$, and $C_\tau^{\alpha,\alpha}(H)$ ($\alpha \in (0, 1)$) of H -valued grid functions $f_\tau = \{f_k\}_{k=1}^{N-1}$ with the corresponding norms,

$$\begin{aligned} \|f_\tau\|_{C_\tau(H)} &= \max_{1 \leq k \leq N-1} \|f_k\|_H, \\ \|f_\tau\|_{C_\tau^\alpha(H)} &= \|f_\tau\|_{C_\tau(H)} + \sup_{1 \leq k < k+n \leq N-1} \frac{\|f_{k+n} - f_k\|_H}{(n\tau)^\alpha}, \\ \|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)} &= \|f_\tau\|_{C_\tau(H)} + \sup_{1 \leq k < k+n \leq N-1} \frac{(k\tau + n\tau)^\alpha (1 - k\tau)^\alpha \|f_{k+n} - f_k\|_H}{(n\tau)^\alpha}. \end{aligned}$$

Theorem 3 *Under the assumptions that $\varphi, \psi, \zeta \in D(A)$, $f_\tau \in C_\tau^{\alpha,\alpha}(H)$ the solution $(p, \{u_k\}_{k=1}^{N-1})$ of the difference problems (28) and (30) in $C_\tau(H) \times H$ obeys the following stability estimates:*

$$\|\{u_k\}_{k=1}^{N-1}\|_{C_\tau(H)} \leq M(\delta, \lambda_1, \dots, \lambda_q) [P_H^0(\varphi, \psi, \zeta) + \|f_\tau\|_{C_\tau(H)}], \tag{31}$$

$$\|A^{-1}p\|_H \leq M(\delta, \lambda_1, \dots, \lambda_q) [P_H^0(\varphi, \psi, \zeta) + \|f_\tau\|_{C_\tau(H)}], \tag{32}$$

$$\|p\|_H \leq M(\delta, \lambda_1, \dots, \lambda_q) \left[P_H^1(\varphi, \psi, \zeta) + \frac{1}{\alpha(1 - \alpha)} \|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)} \right], \tag{33}$$

where $M(\delta, \lambda_1, \dots, \lambda_q)$ does not depend on parameters τ, α , elements φ, ψ, ξ , and the grid function f_τ .

Proof Applying equation (11) at the point $t = t_k$ to the difference problem (28), we get the auxiliary nonlocal difference problem

$$\begin{cases} -\tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Av_k = f_k, & 1 \leq k \leq N - 1, \\ v_1 - v_0 = \tau\varphi, \\ v_N - v_{N-1} = \sum_{i=1}^q k_i(v_{l_i+1} - v_{l_i}) + \tau\psi. \end{cases} \tag{34}$$

By using (25), and the local and nonlocal conditions of problem (34), one can get the system of equations

$$\begin{aligned} (I + R^{2N-1})v_0 - (R^{N-1} + R^N)v_N &= (R - I)^{-1}(I - R^{2N})F_1, \\ \left(-R^{N-1} - R^N - \sum_{i=1}^q k_i(R^{l_i} + R^{2N-l_i-1})\right)v_0 \\ &+ \left((I + R^{2N-1}) + \sum_{i=1}^q k_i(R^{N-l_i-1} + R^{N+l_i})\right)v_N \\ &= -(R - I)^{-1}(I - R^{2N})F_2, \end{aligned} \tag{35}$$

to find v_0 and v_N , where

$$\begin{aligned} F_1 &= \tau\varphi + P(R^{N-1} - R^{N+1})D \sum_{j=1}^{N-1} (R^{N-j} - R^{N+j})f_j\tau - D \sum_{j=1}^{N-1} (R^{1-j} - R^{1+j})f_j\tau, \\ F_2 &= \tau\psi + P \left[R - R^{2N-1} \right. \\ &\quad \left. - \sum_{i=1}^q k_i(R^{N-l_i-1} - R^{N+l_i+1} - R^{N-l_i} + R^{N+l_i}) \right] \\ &\quad \times D \sum_{j=1}^{N-1} (R^{N-j} - R^{N+j})f_j\tau - D \sum_{j=1}^{N-1} \left[R^{|N-1-j|} - R^{N-1+j} \right. \\ &\quad \left. - \sum_{i=1}^q k_i(R^{|l_i+1-j|} - R^{l_i+1+j} - R^{|l_i-j|} + R^{l_i+j}) \right] f_j\tau. \end{aligned} \tag{36}$$

It easy to check that the determinant of system (35) is defined by equation (22). Since the operator Δ_1 has a bounded inverse, we obtain the solution of (35),

$$\begin{aligned} v_0 &= \Delta_1^{-1} \left\{ -(I + R^{2N-1}) + \sum_{i=1}^q k_i(R^{N-l_i-1} + R^{N+l_i})(R - I)^{-1}(I - R^{2N})F_1 \right. \\ &\quad \left. + (R^{N-1} + R^N)(R - I)^{-1}(I - R^{2N})F_2 \right\}, \\ v_N &= \Delta_1^{-1} \left\{ \left(R^{N-1} + R^N + \sum_{i=1}^q k_i(R^{l_i} + R^{2N-l_i-1}) \right) \right. \\ &\quad \left. \times (R - I)^{-1}(I - R^{2N})F_1 - (I + R^{2N-1})(R - I)^{-1}(I - R^{2N})F_2 \right\}, \end{aligned} \tag{37}$$

Hence, problem (34) has a unique solution $\{v_k\}_{k=0}^N$ and it is defined by equations (25), (37).

Applying equation (11) to the difference problems (30), we have the auxiliary nonlocal difference problem

$$\begin{cases} -\tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Av_k = f_k, & 1 \leq k \leq N - 1, \\ \frac{-3v_0 + 4v_1 - v_2}{2\tau} = \varphi, \\ \frac{3v_N - 4v_{N-1} + v_{N-2}}{2\tau} = \psi + \sum_{i=1}^q k_i \left[\frac{3v_{l_i+1} - 4v_{l_i} + v_{l_i-1}}{2\tau} + 2\mu_i \frac{v_{l_i+1} - 2v_{l_i} + v_{l_i-1}}{\tau} \right]. \end{cases} \tag{38}$$

Later, by using the local and nonlocal conditions of problem (38), we have the following system of equations:

$$\begin{aligned} & P(I - R) \{ [R - 3I + R^{2N-2}(I - 3R)]v_0 \\ & \quad + (I + R)R^{N-2}(-R^2 + 4R - I)v_N \} = F_3, \\ & P(I - R) \left\{ \left[(I + R)R^{N-2}(R^2 - 4R + I) \right. \right. \\ & \quad \left. \left. - \sum_{i=1}^q k_i [R^{l_i-1}(I - 3R + \mu_i(I - R)) - R^{2N-l_i-1}(3I - R + \mu_i(I - R))] \right] v_0 \right. \\ & \quad \left. + \left[3I - R - R^{2N-2}(I - 3R) - \sum_{i=1}^q k_i [R^{N-l_i-1}(3I - R + \mu_i(I - R)) \right. \right. \\ & \quad \left. \left. - R^{2N+l_i-1}(I - 3R + \mu_i(I - R))] \right] v_N \right\} = F_4, \end{aligned} \tag{39}$$

where

$$\begin{aligned} F_3 = & 2\tau\varphi + P(I - R)(I + R)R^{N-2}(-R^2 + 4R - I) \\ & - D \sum_{j=1}^{N-1} [4(R^{|N-1-j|} - R^{N-1+j}) + (R^{|N-2-j|} - R^{N-2+j})] f_j \tau, \end{aligned} \tag{40}$$

$$\begin{aligned} F_4 = & 2\tau\psi + P(I - R) \left[3I - R - R^{N-2}(I - 3R) \right. \\ & \left. - \sum_{i=1}^q k_i [R^{N-l_i-1}(3I - R + \mu_i(I - R)) - R^{2N+l_i-1}(I - 3R + \mu_i(I - R))] \right] \\ & - D \sum_{j=1}^{N-1} \left[-4(R^{|1-j|} - R^{1+j}) + (R^{|2-j|} - R^{2+j}) \right. \\ & \left. + \sum_{i=1}^q k_i \left[\left(3 + 2\frac{\mu_i}{\tau} \right) (R^{|l_i+1-j|} - R^{l_i+1+j}) + \left(-4 + 4\frac{\mu_i}{\tau} \right) (R^{|l_i-j|} - R^{l_i+j}) \right. \right. \\ & \left. \left. + \left(1 + 2\frac{\mu_i}{\tau} \right) (R^{|l_i-j-1|} - R^{l_i+j-1}) \right] \right] f_j \tau, \end{aligned} \tag{41}$$

$$\begin{aligned} \Delta_2 = & [R - 3I + R^{2N-2}(I - 3R)] \left[3I - R - R^{2N-2}(I - 3R) \right. \\ & \left. - \sum_{i=1}^q k_i [R^{N-l_i-1}(3I - R + \mu_i(I - R)) - R^{2N+l_i-1}(I - 3R + \mu_i(I - R))] \right] \\ & - (I + R)R^{N-2}(-R^2 + 4R - I) \left[(I + R)R^{N-2}(R^2 - 4R + I) \right. \\ & \left. - \sum_{i=1}^q k_i [R^{l_i-1}(I - 3R + \mu_i(I - R)) - R^{2N-l_i-1}(3I - R + \mu_i(I - R))] \right]. \end{aligned}$$

Since the operator Δ_2 has bounded inverse, we obtain the solution of (39) in the following form:

$$\begin{aligned} v_0 = & \Delta_2^{-1}(I - R)^{-1}(I - R^{2N}) \\ & \times \left\{ \left[3I - R - R^{2N-2}(I - 3R) - \sum_{i=1}^q k_i [R^{N-l_i-1}(3I - R + \mu_i(I - R)) \right. \right. \\ & \left. \left. - R^{2N+l_i-1}(I - 3R + \mu_i(I - R))] \right] F_3 - (I + R)R^{N-2}(-R^2 + 4R - I)F_4 \right\}, \\ v_N = & \Delta_2^{-1}(I - R)^{-1}(I - R^{2N}) \left\{ - \left[(I + R)R^{N-2}(R^2 - 4R + I) \right. \right. \\ & \left. \left. - \sum_{i=1}^q k_i [R^{l_i-1}(I - 3R + \mu_i(I - R)) - R^{2N-l_i-1}(3I - R + \mu_i(I - R))] \right] F_3 \right. \\ & \left. + [R - 3I + R^{2N-2}(I - 3R)]F_4 \right\}. \end{aligned} \tag{42}$$

Thus, problem (34) has a unique solution $\{v_k\}_{k=0}^N$ and it is defined by equations (25), (42). Applying equations (25), (37), (42) and the method of [28], we get

$$\| \{v_k\}_{k=1}^{N-1} \|_{C_\tau(H)} \leq M(\delta, \lambda_1, \dots, \lambda_q) [P_H^0(\varphi, \psi, \zeta) + \|f_\tau\|_{C_\tau(H)}], \tag{43}$$

$$\begin{aligned} & \| \{ \tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) \}_{k=1}^{N-1} \|_{C_\tau^{\alpha,\alpha}(H)} + \| \{Av_k\}_{k=1}^{N-1} \|_{C_\tau^{\alpha,\alpha}(H)} \\ & \leq M(\delta, \lambda_1, \dots, \lambda_q) \left[P_H^1(\varphi, \psi, \zeta) + \frac{1}{\alpha(1-\alpha)} \|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)} \right], \end{aligned} \tag{44}$$

for the solutions of both difference problems (34), (38). The proofs of the estimates (32), (33) for the solutions of the difference problems (34), (38) are based on equation (11) and estimates (43), (44). By using equation (11) and estimates (43), (32), we can get estimate (31). \square

Denote by F an SAPD operator in a Hilbert space H , introduce $E_\alpha = E_\alpha(D(F), H)$, the Banach space of such functions $u \in H$ for which the norm

$$\|u\|_{E_\alpha} = \sup_{z>0} z^{1-\alpha} \|Fe^{-zF}u\|_H + \|u\|_H$$

is finite.

Theorem 4 Under the assumptions $\varphi, \psi, \zeta \in D(C), f_\tau \in C_\tau^{\alpha, \alpha}(H)$ ($\alpha \in (0, 1)$) the solution $(p, \{u_k\}_{k=1}^{N-1})$ of the difference problems (28) and (30) obeys the coercive stability estimate

$$\begin{aligned} & \left\| \left\{ \frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C_\tau(H)} + \left\| \{Au_k\}_{k=1}^{N-1} \right\|_{C_\tau(H)} + \|p\|_H \\ & \leq M(\delta, \lambda_1, \dots, \lambda_q) \left[\|C\varphi\|_{E_\alpha} + \|C\psi\|_{E_\alpha} + \|C\zeta\|_{E_\alpha} + \frac{1}{\alpha(1-\alpha)} \|f_\tau\|_{C_\tau(H)} \right], \end{aligned} \tag{45}$$

where $M(\delta, \lambda_1, \dots, \lambda_q)$ is independent from parameters τ, α , elements φ, ψ, ξ , and the grid function f_τ .

4 Multidimensional elliptic problem

Now, we give the application of the abstract Theorems 1 and 3.

Let $\Omega = (0, \ell) \times \dots \times (0, \ell)$ be the open cube in the n -dimensional Euclidean space with boundary $S, \overline{\Omega} = \Omega \cup S$ and numbers $\lambda_0, \lambda_i, i = 1, \dots, q$ ($0 < \lambda_1 < \lambda_2 < \dots < \lambda_q < T$) and nonnegative real coefficients $k_i, i = 1, \dots, q$, be given, condition (2) for coefficients be valid, $x = (x_1, \dots, x_n)$. Consider the inverse problem of finding functions $p(x)$ and $u(t, x)$ for the multidimensional elliptic equation in $[0, T] \times \Omega$,

$$\begin{cases} -u_{tt}(t, x) - \sum_{r=1}^n (a_r(x)u_{x_r})_{x_r} + \sigma u(t, x) = f(t, x) + p(x), \\ 0 < t < T, x \in \Omega, \\ u_t(0, x) = \varphi(x), \quad u_t(T, x) = \sum_{i=1}^q k_i u_t(\lambda_i, x) + \psi(x), \\ u(\lambda_0, x) = \zeta(x), \quad x \in \overline{\Omega}, \\ u(t, x) = 0, \quad x \in S, 0 \leq t \leq T. \end{cases} \tag{46}$$

Here, $\sigma > 0$ is a known number, $a_r(x)$ ($x \in \Omega$), $\varphi(x), \psi(x), \zeta(x)$ ($x \in \overline{\Omega}$), and $f(t, x)$ ($t \in (0, 1), x \in \Omega$) are given smooth functions, $a_r(x) \geq a > 0$ ($x \in \Omega$).

It is well known that the differential expression [28]

$$A^x u(x) = - \sum_{r=1}^n (a_r(x)u_{x_r})_{x_r} + \sigma u(x) \tag{47}$$

defines a self-adjoint positive definite operator A^x acting on $L_2(\overline{\Omega})$ with the domain $D(A^x) = \{u(x) \in W_2^2(\overline{\Omega}), u = 0 \text{ on } S\}$.

Let H be the Hilbert space $L_2(\overline{\Omega})$. Denote by $C_{0T}^{\alpha, \alpha}(L_2(\overline{\Omega}))$ the space obtained by completion of the space of all smooth $L_2(\overline{\Omega})$ -valued functions ρ on $[0, 1]$ with the norm

$$\|\rho\|_{C_{0T}^{\alpha, \alpha}(L_2(\overline{\Omega}))} = \|\rho\|_{C(L_2(\overline{\Omega}))} + \sup_{0 \leq t < \tau \leq 1} \frac{(t + \tau)^\alpha (1 - t)^\alpha \|\rho(t + \tau) - \rho(t)\|_{L_2(\overline{\Omega})}}{\tau^\alpha}.$$

Applying the abstract Theorems 1 and 3, we get the following estimates for the solution of problem (46).

Theorem 5 Assume that A^x is defined by equation (47), condition (2) for coefficients is valid, $f \in C_{0T}^{\alpha, \alpha}(L_2(\overline{\Omega}))$ and $\varphi, \zeta, \psi \in D(A^x) \cap D((A^x)^{-\frac{1}{2}})$. Then, for the solutions (p, u) of the

inverse boundary value problem (46), the stability estimates

$$\begin{aligned} \|u\|_{C(L_2(\bar{\Omega}))} &\leq M(\delta) \left[P_{L_2(\bar{\Omega})}^0(\varphi, \psi, \zeta) + P_{L_2(\bar{\Omega})}^{-\frac{1}{2}}(\varphi, \psi, \zeta) + \|f\|_{C(L_2(\bar{\Omega}))} \right], \\ \|(A^x)^{-1}p\|_{L_2(\bar{\Omega})} &\leq M(\delta) \left[P_{L_2(\bar{\Omega})}^0(\varphi, \psi, \zeta) + \|f\|_{C(L_2(\bar{\Omega}))} \right], \\ \|p\|_{L_2(\bar{\Omega})} &\leq M(\delta) \left[P_{L_2(\bar{\Omega})}^1(\varphi, \psi, \zeta) + \frac{1}{\alpha(1-\alpha)} \|f\|_{C_{0T}^{\alpha,\alpha}(L_2(\bar{\Omega}))} \right], \end{aligned}$$

are satisfied, where $M(\delta)$ is independent of $\alpha, \varphi(x), \zeta(x), \psi(x)$, and $f(t, x)$.

Now, we will discretize problem (46) into two steps. In the first step, we define the grid spaces

$$\begin{aligned} \tilde{\Omega}_h &= \{x = x_m = (h_1 m_1, \dots, h_n m_n); m = (m_1, \dots, m_n), \\ &\quad m_r = 0, \dots, M_r, h_r M_r = \ell, r = 1, \dots, n\}, \\ \Omega_h &= \tilde{\Omega}_h \cap \Omega, \quad S_h = \tilde{\Omega}_h \cap S. \end{aligned}$$

To the differential operator A^x generated by problem (46) we assign the difference operator A_h^x defined by the formula

$$A_h^x u^h(x) = - \sum_{r=1}^n (a_r(x) u_{\tilde{x}_r}^h)_{x_r, j_r}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$. It is well known that A_h^x is a self-adjoint positive definite operator.

By using A_h^x , we arrive at the following BVP:

$$\begin{cases} -\frac{d^2 u^h(t, x)}{dt^2} + A_h^x u^h(t, x) = f^h(t, x) + p^h(x), & 0 < t < T, x \in \Omega_h, \\ \frac{du^h}{dt}(0, x) = \varphi^h(x), \\ \frac{du^h}{dt}(T, x) - \sum_{i=1}^q k_i \frac{du^h}{dt}(\lambda_i, x) = \psi^h(x), & x \in \tilde{\Omega}_h, \end{cases} \tag{48}$$

for a system of ordinary differential equations.

In the second step of discretization, problem (48) is replaced by the following difference schemes:

$$\begin{cases} -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = f_k^h(x) + p^h(x), \\ f_k^h(x) = f^h(t_k, x), \quad t_k = k\tau, 1 \leq k \leq N-1, x \in \Omega_h, \\ u_1^h(x) - u_0^h(x) = \tau \varphi^h(x), \\ u_N^h(x) - u_{N-1}^h(x) - \sum_{i=1}^q k_i (u_{i+1}^h(x) - u_i^h(x)) = \tau \psi^h(x), \\ u_{l_0}^h(x) = \zeta^h(x), \quad x \in \tilde{\Omega}_h, \end{cases} \tag{49}$$

$$\begin{cases}
 -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = f_k^h(x) + p^h(x), \\
 f_k^h(x) = f^h(t_k, x), \quad t_k = k\tau, 1 \leq k \leq N, x \in \Omega_h, \\
 -3u_0^h(x) + 4u_1^h(x) - 2u_2^h(x) = 2\tau\varphi^h(x), \\
 3u_N^h(x) - 4u_{N-1}^h(x) + u_{N-2}^h(x) \\
 \quad - \sum_{i=1}^q k_i [(3 + 2\mu_i)u_{i+1}^h(x) - (4 + 4\mu_i)u_i^h(x) + (1 + 2\mu_i)u_{i-1}^h(x)] \\
 = 2\tau\psi^h(x), u_{l_0}^h(x) + \mu_i(u_{l_0+1}^h(x) - u_{l_0}^h(x)) = \zeta(x), \quad x \in \tilde{\Omega}_h.
 \end{cases} \tag{50}$$

For the calculation of $p^h(x)$ we have

$$p^h(x) = A_h^x \zeta^h(x) - A_h^x v^h(\lambda_0, x), \quad x \in \tilde{\Omega}_h. \tag{51}$$

To formulate our results, let $L_{2h} = L_2(\tilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\tilde{\Omega}_h)$ be spaces of the grid functions $\rho^h(x) = \{\rho(h_1 m_1, \dots, h_n m_n)\}$ defined on $\tilde{\Omega}_h$, equipped with the norms

$$\begin{aligned}
 \|\rho^h\|_{L_{2h}} &= \left(\sum_{x \in \tilde{\Omega}_h} |\rho^h(x)|^2 h_1 \cdots h_n \right)^{1/2}, \\
 \|\rho^h\|_{W_{2h}^2} &= \|\rho^h\|_{L_{2h}} + \left(\sum_{x \in \tilde{\Omega}_h} \sum_{r=1}^n |(\rho^h)_{x_r}|^2 h_1 \cdots h_n \right)^{1/2} \\
 &\quad + \left(\sum_{x \in \tilde{\Omega}_h} \sum_{r=1}^n |(\rho^h(x))_{x_r, \bar{x}_r, m_r}|^2 h_1 \cdots h_n \right)^{1/2}.
 \end{aligned}$$

Let τ and $|h| = \sqrt{h_1^2 + \dots + h_n^2}$ be sufficiently small positive numbers.

Theorem 6 *For the solutions of the difference schemes (49) and (50) the following stability estimates hold:*

$$\begin{aligned}
 \|\{u_k^h\}_1^{N-1}\|_{C_\tau(L_{2h})} &\leq M(\delta, \lambda_1, \dots, \lambda_q) [\|\varphi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}} \\
 &\quad + \|\zeta^h\|_{L_{2h}} + \|\{f_k^h\}_1^{N-1}\|_{C_\tau(L_{2h})}], \\
 \|p^h\|_{L_{2h}} &\leq M(\delta, \lambda_1, \dots, \lambda_q) \left[\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|\zeta^h\|_{W_{2h}^2} \right. \\
 &\quad \left. + \frac{1}{\alpha(1-\alpha)} \|\{f_k^h\}_1^{N-1}\|_{C_\tau(L_{2h})} \right],
 \end{aligned}$$

where $M(\delta, \lambda_1, \dots, \lambda_q)$ is independent of $\tau, \alpha, h, \varphi^h(x), \psi^h(x), \zeta^h(x)$ and $\{f_k^h(x)\}_1^{N-1}$.

The proof of Theorem 6 is based on the symmetry property of the operator A_h^x in L_{2h} and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h} .

Theorem 7 ([30]) *For the solution of the elliptic difference problem*

$$\begin{cases}
 A_h^x u^h(x) = \omega^h(x), \quad x \in \tilde{\Omega}_h, \\
 u^h(x) = 0, \quad x \in S_h,
 \end{cases} \tag{52}$$

the following coercivity inequality holds:

$$\sum_{r=1}^n \|(u_k^h)_{\bar{x}_r, \bar{x}_r, j_r}\|_{L_{2h}} \leq M \|\omega^h\|_{L_{2h}},$$

where M is independent of h and ω .

5 Numerical results

In this section, we present numerical results with explanation on the realization for two dimensional and three dimensional examples of the Bitsadze-Samarskii type overdetermined elliptic multipoint NBVP. The MATLAB program is used to get numerical results.

5.1 Two dimensional example

It is easy to check that pair functions $p(x) = [(1+x)\pi^2 + 1] \sin(\pi x) - \pi \cos(\pi x)$ and $u(t, x) = (\exp(-t) + 1) \sin(\pi x)$ is an exact solution of the following two dimensional overdetermined elliptic three-point NBVP:

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2}(t, x) - \frac{\partial}{\partial x}((1+x)\frac{\partial u}{\partial x}(t, x)) + u(t, x) = f(t, x) + p(x), \\ 0 < x < 1, 0 < t < 1, \\ u_t(1, x) - \frac{1}{4}u_t(\frac{1}{4}, x) - \frac{1}{4}u_t(\frac{1}{2}, x) - \frac{1}{4}u_t(\frac{3}{4}, x) = \psi(x), \\ u_t(0, x) = \varphi(x), \quad u(\frac{2}{5}, x) = \zeta(x), \quad 0 \leq x \leq 1, \\ u(t, 0) = u(t, 1) = 0, \quad 0 \leq t \leq 1, \end{cases} \tag{53}$$

where

$$\begin{aligned} f(t, x) &= (1+x)\pi^2 e^{-t} \sin(\pi x) - \pi e^{-t} \cos(\pi x), \\ \psi(x) &= \left[-e^{-1} + \frac{1}{4}(e^{-\frac{1}{4}} + e^{-\frac{1}{2}} + e^{-\frac{3}{4}}) \right] \sin(\pi x), \\ \zeta(x) &= (e^{-\frac{2}{5}} + 1) \sin(\pi x). \end{aligned}$$

Denote by $[0, 1]_\tau \times [0, 1]_h$ the set of grid points depending on the small parameters τ and h

$$[0, 1]_\tau \times [0, 1]_h = \{(t_k, x_n) : t_k = k\tau, k = \overline{0, N}, x_n = nh, n = \overline{0, M}\},$$

where $N\tau = 1, Mh = 1$. Let us

$$\begin{aligned} \lambda_0 &= \frac{2}{5}, \quad \lambda_1 = \frac{1}{4}, \quad \lambda_2 = \frac{1}{2}, \quad \lambda_3 = \frac{3}{4}; \\ l_i &= \left\lceil \frac{\lambda_i}{\tau} \right\rceil, \quad \mu_i = \frac{\lambda_i}{\tau} - l_i, \quad i = 0, 1, 2, 3; \\ \varphi_n &= \varphi(x_n), \quad \psi_n = \psi(x_n), \quad \zeta_n = \zeta(x_n), \quad n = \overline{0, M}; \\ f_n^k &= f(t_k, x_n), \quad k = \overline{0, N}, n = \overline{0, M}. \end{aligned}$$

To approximately solve (53), we use an algorithm with three stages. In the first stage, we find a numerical solution of the auxiliary NBVP. We can write

$$\begin{cases} \frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} + (1 + x_n) \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + \frac{v_{n+1}^k - v_{n-1}^k}{2h} = f(t_k, x_n), \\ k = \overline{1, N-1}, n = \overline{1, M-1}, \\ v_0^k = v_M^k = 0, \quad k = \overline{0, N}, \quad v_n^1 - v_n^0 = \tau \varphi_n, \\ v_n^N - v_n^{N-1} - \frac{1}{4}(v_n^{1+1} - v_n^{1-1}) - \frac{1}{4}(v_n^{2+1} - v_n^{2-1}) - \frac{1}{4}(v_n^{3+1} - v_n^{3-1}) = \psi_n, \quad n = \overline{0, M}, \end{cases} \tag{54}$$

the first order of accuracy in t and the second order of accuracy in x and

$$\begin{cases} \frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} + (1 + x_n) \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + \frac{v_{n+1}^k - v_{n-1}^k}{2h} = f(t_k, x_n), \\ k = \overline{1, N-1}, n = \overline{1, M-1}, \\ v_0^k = v_M^k = 0, \quad k = \overline{0, N}, \quad -3v_n^0 + 4v_n^1 - v_n^2 = 2\tau \varphi_n, \\ 3v_n^N - 4v_n^{N-1} + v_n^{N-2} - \frac{1}{4}[(3 + 2\mu_1)v_n^{1+1} - (4 + 4\mu_1)v_n^{1-1} + (1 + 2\mu_1)v_n^{1-1}] \\ - \frac{1}{4}[(3 + 2\mu_2)v_n^{2+1} - (4 + 4\mu_2)v_n^{2-1} + (1 + 2\mu_2)v_n^{2-1}] \\ - \frac{1}{4}[(3 + 2\mu_3)v_n^{3+1} - (4 + 4\mu_3)v_n^{3-1} + (1 + 2\mu_3)v_n^{3-1}] = 2\tau \psi_n, \quad n = \overline{0, M}, \end{cases} \tag{55}$$

the second order of accuracy in t and the second order of accuracy in x difference schemes for corresponding NBVP.

In the second stage, we find p_n . It is carried out by

$$p_n = -(1 + x_n) \frac{(\zeta_{n+1} - v_{n+1}^{l_0}) - 2(\zeta_n - v_n^{l_0}) + (\zeta_{n-1} - v_{n-1}^{l_0})}{h^2} - \frac{(\zeta_{n+1} - v_{n+1}^{l_0}) - (\zeta_{n-1} - v_{n-1}^{l_0})}{2h} + v_n^{l_0}, \quad n = \overline{1, M-1},$$

for the first order approximation, and

$$\begin{aligned} p_n = & -\frac{1 + x_n}{h^2} \{ [\zeta_{n+1} - (\mu_0 v_{n+1}^{l_0+1} - (\mu_0 - 1)v_{n+1}^{l_0})] \\ & - 2[\zeta_n - (\mu_0 v_n^{l_0+1} - (\mu_0 - 1)v_n^{l_0})] \\ & + [\zeta_{n-1} - (\mu_0 v_{n-1}^{l_0+1} - (\mu_0 - 1)v_{n-1}^{l_0})] \} \\ & - \frac{1}{2h} \{ [\zeta_{n+1} - (\mu_0 v_{n+1}^{l_0+1} - (\mu_0 - 1)v_{n+1}^{l_0})] \\ & - [\zeta_{n-1} - (\mu_0 v_{n-1}^{l_0+1} - (\mu_0 - 1)v_{n-1}^{l_0})] \} \\ & + \zeta_n - (\mu_0 v_n^{l_0+1} - (\mu_0 - 1)v_n^{l_0}), \quad n = \overline{1, M-1}, \end{aligned}$$

for the second order approximation.

Both difference problems (54) and (55) can be rewritten in the matrix form

$$\begin{aligned} A_n v_{n+1} + B_n v_n + C_n v_{n-1} &= I g_n, \quad n = \overline{1, M-1}, \\ v_0 &= \vec{0}, \quad v_M = \vec{0}. \end{aligned} \tag{56}$$

Here, g_n is $(N + 1) \times 1$ a column matrix, A_n, B_n, C_n are $(N + 1) \times (N + 1)$ square matrices, and I is the $(N + 1) \times (N + 1)$ identity matrix, v_s is the $(N + 1) \times 1$ matrix $v_s = [v_s^0 \dots v_s^N]^t$, $s = n - 1, n, n + 1$. Denote

$$a_n = \frac{1 + x_n}{h^2} + \frac{1}{2h}, \quad c_n = \frac{1 + x_n}{h^2} - \frac{1}{2h}, \quad b_n = -\frac{2}{\tau^2} - \frac{2(1 + x_n)}{h^2}, \quad d = \frac{1}{\tau^2}.$$

Then

$$\begin{aligned} A_n &= \text{diag}(0, a_n, a_n, \dots, a_n, 0), & C_n &= \text{diag}(0, c_n, c_n, \dots, c_n, 0), \\ g_n^k &= -f(t_k, x_n), & k &= \overline{1, N-1}, n = \overline{1, M-1}, \\ B_n[i, i] &= b_n, & B_n[i-1, i] &= d, & B_n[i, i-1] &= d, & i &= \overline{2, N}, \\ B_n[1, 1] &= -1, & B_n[1, 2] &= 1, & B_n[N+1, N+1] &= 1, \\ B_n[N+1, N] &= -1, \\ B_n[N+1, l_1] &= -\frac{1}{4}, & B_n[N+1, l_2] &= -\frac{1}{4}, & B_n[N+1, l_3] &= -\frac{1}{4} \\ B_n[N+1, l_1+1] &= \frac{1}{4}, & B_n[N+1, l_2+1] &= \frac{1}{4}, \\ B_n[N+1, l_3+1] &= \frac{1}{4}, \\ B_n[i, j] &= 0, & \text{for other } i \text{ and } j, \\ g_n^0 &= \tau \varphi_n, & g_n^N &= \tau \psi_n, & n &= \overline{1, M-1}, \end{aligned}$$

for the first order approximation, and

$$\begin{aligned} B_n[i, i] &= b_n, & B_n[i-1, i] &= d, & B_n[i, i-1] &= d, & i &= \overline{2, N}, \\ B_n[1, 1] &= -3, & B_n[1, 2] &= 4, & B_n[1, 3] &= -1, & B_n[N+1, N+1] &= 3, \\ B_n[N+1, N] &= -4, & B_n[N+1, N-1] &= -1, \\ B_n[N+1, l_1+1] &= -\frac{1}{4}(3 + 2\mu_1), & B_n[N+1, l_1+1] &= \frac{1}{4}(3 + 2\mu_1), \\ B_n[N+1, l_1-1] &= -\frac{1}{4}(1 + 2\mu_1), & B_n[N+1, l_2+1] &= -\frac{1}{4}(3 + 2\mu_2), \\ B_n[N+1, l_2+1] &= \frac{1}{4}(3 + 2\mu_2), & B_n[N+1, l_2-1] &= -\frac{1}{4}(1 + 2\mu_2), \\ B_n[N+1, l_3+1] &= -\frac{1}{4}(3 + 2\mu_3), & B_n[N+1, l_3+1] &= \frac{1}{4}(3 + 2\mu_3), \\ B_n[N+1, l_3-1] &= -\frac{1}{4}(1 + 2\mu_3), \\ B_n[i, j] &= 0, & \text{for other } i \text{ and } j, \\ g_n^0 &= 2\tau \varphi_n, & g_n^N &= 2\tau \psi_n, & n &= \overline{1, M-1}, \end{aligned}$$

for the second order approximation.

Finally, in the third stage, we calculate $\{u_n^k\}$ by $u_n^k = v_n^k + \zeta_n - v_n^{l_0}$, and $u_n^k = v_n^k + \zeta_n - (\mu_0 v_n^{l_0+1} - (\mu_0 - 1)v_n^{l_0})$, for the first and second order approximations, respectively.

Table 1 Error of v in example (53)

Difference scheme	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
First order of the ADS	6.12×10^{-3}	2.69×10^{-3}	1.25×10^{-3}	6.03×10^{-4}
Second order of the ADS	1.73×10^{-3}	4.36×10^{-4}	1.09×10^{-4}	2.73×10^{-5}

Table 2 Error of u in example (53)

Difference scheme	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
First order of the ADS	3.78×10^{-3}	1.78×10^{-3}	8.63×10^{-4}	4.24×10^{-4}
Second order of the ADS	3.88×10^{-4}	9.74×10^{-5}	2.44×10^{-5}	6.10×10^{-6}

Table 3 Error of p in example (53)

Type of approximation	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
First order	0.022091	0.010516	0.0051328	0.002536
Second order	3.64×10^{-4}	9.48×10^{-5}	2.40×10^{-5}	6.05×10^{-6}

To solve (56), we use a modification of the Gauss elimination method ([31]). Let $v_M = \vec{0}$, α_n ($n = 1, \dots, M - 1$) be $(N + 1) \times (N + 1)$ square matrices and β_n ($n = 1, \dots, M - 1$) be $(N + 1) \times 1$ column vectors, α_1 be the zero matrix and β_1 be zero column vector. Searching the solution of (56) by

$$v_n = \alpha_{n+1}v_{n+1} + \beta_{n+1}, \quad n = M - 1, \dots, 1,$$

we get formulas for $\alpha_{n+1}, \beta_{n+1}$:

$$\begin{aligned} \alpha_{n+1} &= -(B_n + C_n\alpha_n)^{-1}A_n, \\ \beta_{n+1} &= -(B_n + C_n\alpha_n)^{-1}(I\theta_n - C_n\beta_n), \quad n = 1, \dots, M - 1. \end{aligned}$$

Results of numerical calculations are presented in Tables 1-3 for both the first and second order approximations in the cases $N = M = 20, 40, 80$ and 160. Table 1 gives the error between the exact solution of NBVP and the solutions derived by difference schemes. Table 2 contains the error between exact and approximate u . Tables 3 includes the error for p . It can be seen from Tables 1-3 that the second order of the ADS is more accurate compared to the first order of the ADS.

5.2 Three dimensional example

Now, consider the three dimensional overdetermined elliptic two point NBVP

$$\left\{ \begin{aligned} &-\frac{\partial^2 u}{\partial t^2}(t, x, y) - \frac{\partial^2 u}{\partial x^2}(t, x, y) - \frac{\partial^2 u}{\partial y^2}(t, x, y) + u(t, x, y) = f(t, x, y) + p(x, y), \\ &0 < x < 1, 0 < y < 1, 0 < t < 1, \\ &u_t(0, x, y) = \varphi(x, y), \\ &u_t(1, x, y) - \frac{1}{3}u_t(\frac{1}{3}, x, y) - \frac{1}{3}u_t(\frac{2}{3}, x, y) = \psi(x, y), \\ &u(\frac{3}{5}, x, y) = \zeta(x, y), \\ &0 \leq x \leq 1, 0 \leq y \leq 1, u(t, 0) = u(t, 1) = 0, 0 \leq t \leq 1, \end{aligned} \right. \tag{57}$$

where

$$\begin{aligned}
 f(t, x, y) &= 2\pi^2 e^{-t} \sin(\pi x) \sin(\pi y), \\
 \varphi(x, y) &= -\sin(\pi x) \sin(\pi y), \\
 \psi(x, y) &= \left[-e^{-1} + \frac{1}{3} (e^{-\frac{1}{3}} + e^{-\frac{2}{3}}) \right] \sin(\pi x) \sin(\pi y), \\
 \zeta(x, y) &= (e^{-\frac{3}{5}} + 1) \sin(\pi x) \sin(\pi y).
 \end{aligned}$$

It is easy to see that the pair functions $p(x, y) = (2\pi^2 + 1) \sin(\pi x) \sin(\pi y)$ and $u(t, x, y) = (e^{-t} + 1) \sin(\pi x) \sin(\pi y)$ are an exact solution of (57).

Denote by $[0, 1]_\tau \times [0, 1]_h \times [0, 1]_h$ the set of grid points depending on the small parameters τ and h ,

$$\begin{aligned}
 [0, 1]_\tau \times [0, 1]_h \times [0, 1]_h &= \{(t_k, x_n, y_m) : t_k = k\tau, k = \overline{0, N}, \\
 &x_n = nh, n = \overline{0, M}, y_m = mh, m = \overline{0, M}, N\tau = 1, Mh = 1\}.
 \end{aligned}$$

Let us take

$$\begin{aligned}
 \lambda_0 &= \frac{3}{5}, \quad \lambda_1 = \frac{1}{4}, \quad \lambda_2 = \frac{3}{4}; \\
 l_i &= \left[\frac{\lambda_i}{\tau} \right], \quad \mu_i = \frac{\lambda_i}{\tau} - l_i, \quad i = 0, 1, 2; \\
 \varphi_{m,n} &= \varphi(x_n, y_m), \quad \psi_{m,n} = \psi(x_n, y_m), \quad \zeta_{m,n} = \zeta(x_n, y_m), \\
 n &= \overline{0, M}, m = \overline{0, M}; \\
 f_{m,n}^k &= f(t_k, x_n, y_m), \quad k = \overline{0, N}, n = \overline{0, M}, m = \overline{0, M}.
 \end{aligned}$$

In the first stage, the difference schemes for the approximate solution of NBVP can be written in the following forms:

$$\left\{ \begin{aligned}
 &-\frac{v_{m,n}^{k+1} - 2v_{m,n}^k + v_{m,n}^{k-1}}{\tau^2} - \frac{v_{m,n+1}^k - 2v_{m,n}^k + v_{m,n-1}^k}{h^2} \\
 &\quad - \frac{v_{m+1,n}^k - 2v_{m,n}^k + v_{m-1,n}^k}{\tau^2} + v_{m,n}^k = f_{m,n}^k, \\
 &k = \overline{1, N-1}, n = \overline{1, M-1}, m = \overline{1, M-1}, \\
 &v_{0,n}^k = v_{M,n}^k = v_{m,n}^k = v_{m,M}^k = 0, \\
 &k = 0, \dots, N, n = \overline{1, M-1}, m = \overline{1, M-1}, \\
 &v_{m,n}^1 - v_{m,n}^0 = \tau \varphi_{m,n}, \\
 &v_{m,n}^N - v_{m,n}^{N-1} - \frac{1}{3} (v_{m,n}^{l_1+1} - v_{m,n}^{l_1}) - \frac{1}{3} (v_{m,n}^{l_2+1} - v_{m,n}^{l_2}) = \psi_{m,n}, \\
 &n = \overline{1, M-1}, m = \overline{1, M-1},
 \end{aligned} \right. \tag{58}$$

and

$$\left\{ \begin{array}{l}
 -\frac{v_{m,n}^{k+1}-2v_{m,n}^k+v_{m,n}^{k-1}}{\tau^2} - \frac{v_{m,n+1}^k-2v_{m,n}^k+v_{m,n-1}^k}{h^2} \\
 -\frac{v_{m+1,n}^k-2v_{m,n}^k+v_{m-1,n}^k}{\tau^2} + v_{m,n}^k = f_{m,n}^k, \\
 k = \overline{1, N-1}, n = \overline{1, M-1}, m = \overline{1, M-1}, \\
 v_{0,n}^k = v_{M,n}^k = v_{m,n}^k = v_{m,M}^k = 0, \\
 k = 0, \dots, N, n = \overline{1, M-1}, m = \overline{1, M-1}, \\
 -3v_{m,n}^0 + 4v_{m,n}^1 - v_{m,n}^2 \\
 = 2\tau\varphi_{m,n}, 3v_{m,n}^N - 4v_{m,n}^{N-1} + v_{m,n}^{N-2} \\
 -\frac{1}{3}[(3 + 2\mu_1)v_{m,n}^{l_1+1} - (4 + 4\mu_1)v_{m,n}^{l_1} + (1 + 2\mu_1)v_{m,n}^{l_1}] \\
 -\frac{1}{3}[(3 + 2\mu_2)v_{m,n}^{l_2+1} - (4 + 4\mu_2)v_{m,n}^{l_2} + (1 + 2\mu_2)v_{m,n}^{l_2}] = 2\tau\psi_{m,n}, \\
 n = \overline{1, M-1}, n = \overline{1, M-1},
 \end{array} \right. \tag{59}$$

respectively.

In the second stage, the calculation of p_n ($n = \overline{1, M-1}, m = \overline{1, M-1}$) is carried out by

$$p_{m,n} = -\frac{(\zeta_{m,n+1} - v_{m,n+1}^{l_0}) - 2(\zeta_{m,n} - v_{m,n}^{l_0}) + (\zeta_{m,n-1} - v_{m,n-1}^{l_0})}{h^2} \\
 -\frac{(\zeta_{m+1,n} - v_{m+1,n}^{l_0}) - 2(\zeta_{m,n} - v_{m,n}^{l_0}) + (\zeta_{m-1,n} - v_{m-1,n}^{l_0})}{h^2} + v_{m,n}^{l_0},$$

for the first order approximation, and

$$p_{m,n} = -\frac{1}{h^2} \{ [\zeta_{m,n+1} - (\mu_0 v_{m,n+1}^{l_0+1} - (\mu_0 - 1)v_{m,n+1}^{l_0})] \\
 - 2[\zeta_{m,n} - (\mu_0 v_{m,n}^{l_0+1} - (\mu_0 - 1)v_{m,n}^{l_0})] \\
 + [\zeta_{m,n-1} - (\mu_0 v_{m,n-1}^{l_0+1} - (\mu_0 - 1)v_{m,n-1}^{l_0})] \} \\
 -\frac{1}{h^2} \{ [\zeta_{m+1,n} - (\mu_0 v_{m+1,n}^{l_0+1} - (\mu_0 - 1)v_{m+1,n}^{l_0})] \\
 - 2[\zeta_{m,n} - (\mu_0 v_{m,n}^{l_0+1} - (\mu_0 - 1)v_{m,n}^{l_0})] \\
 + [\zeta_{m-1,n} - (\mu_0 v_{m-1,n}^{l_0+1} - (\mu_0 - 1)v_{m-1,n}^{l_0})] \}$$

for the second order approximation.

In the third stage, we calculate $\{u_n^k\}$ by

$$u_{m,n}^k = v_{m,n}^k + \zeta_n - v_{m,n}^{l_0}, \quad \text{and} \quad u_{m,n}^k = v_{m,n}^k + \zeta_{m,n} - (\mu_0 v_{m,n}^{l_0+1} - (\mu_0 - 1)v_{m,n}^{l_0})$$

for the first and second order approximation, respectively.

The difference problems (58) and (59) can be rewritten in the matrix form (56). In this case, g_n is $(N + 1)(M + 1) \times 1$ a column matrix, A, B, C, I are $(N + 1)(M + 1) \times (N + 1)(M + 1)$ square matrices, and I is the identity matrix, v_s is the $(N + 1)(M + 1) \times 1$ column matrix such that

$$v_s = \left[v_{0,s}^0 \quad \dots \quad v_{0,s}^N \quad v_{1,s}^0 \quad \dots \quad v_{1,s}^N \quad \dots \quad v_{M,s}^0 \quad \dots \quad v_{M,s}^N \right]^t, \\
 s = n - 1, n, n + 1.$$

Denote

$$a = \frac{1}{h^2}, \quad b = 1 + \frac{2}{\tau^2} + \frac{4}{h^2}, \quad d = \frac{1}{\tau^2}.$$

Then

$$A = C = \begin{bmatrix} O & O & \dots & O & O \\ O & E & \dots & O & O \\ \dots & \dots & \ddots & \dots & \dots \\ O & O & \dots & E & \\ O & O & \dots & O & O \end{bmatrix}, \quad B = \begin{bmatrix} Q & O & \dots & O & O \\ O & D & \dots & O & O \\ \dots & \dots & \ddots & \dots & \dots \\ O & O & \dots & D & \\ O & O & \dots & O & Q \end{bmatrix},$$

$$E = \text{diag}(0, a, a, \dots, a, 0), \quad Q = I_{(N+1) \times (N+1)}, \quad O = O_{(N+1) \times (N+1)},$$

$$g_{m,n}^k = -f(t_k, x_n, y_m), \quad k = \overline{1, N-1}, n = \overline{1, M-1}, m = \overline{1, M-1},$$

$$D[i, i] = b, \quad D[i-1, i] = d, \quad D[i, i-1] = d, \quad i = \overline{2, N},$$

$$D[1, 1] = -1, \quad D[1, 2] = 1, \quad D[N+1, N+1] = 1,$$

$$D[N+1, N] = -1,$$

$$D[N+1, l_1] = -\frac{1}{4}, \quad D[N+1, l_2] = -\frac{1}{4}, \quad D[N+1, l_3] = -\frac{1}{4}$$

$$D[N+1, l_1+1] = \frac{1}{4}, \quad D[N+1, l_2+1] = \frac{1}{4},$$

$$D[N+1, l_3+1] = \frac{1}{4},$$

$$D[i, j] = 0, \quad \text{for other } i \text{ and } j,$$

$$g_{m,n}^0 = \tau \varphi_{m,n}, \quad g_{m,n}^N = \tau \psi_{m,n}, \quad n = \overline{1, M-1}, m = \overline{1, M-1},$$

for the first order approximation, and

$$D[i, i] = b, \quad D[i-1, i] = d, \quad D[i, i-1] = d, \quad i = \overline{2, N},$$

$$D[1, 1] = -3, \quad D[1, 2] = 4, \quad D[1, 3] = -1, \quad D[N+1, N+1] = 3,$$

$$D[N+1, N] = -4, \quad D[N+1, N-1] = -1,$$

$$D[N+1, l_1+1] = -\frac{1}{3}(3+2\mu_1), \quad D[N+1, l_1+1] = \frac{1}{3}(3+2\mu_1),$$

$$D[N+1, l_1-1] = -\frac{1}{3}(1+2\mu_1), \quad D[N+1, l_2+1] = -\frac{1}{3}(3+2\mu_2),$$

$$D[N+1, l_2+1] = \frac{1}{3}(3+2\mu_2), \quad D[N+1, l_2-1] = -\frac{1}{3}(1+2\mu_2),$$

$$D[i, j] = 0, \quad \text{for other } i \text{ and } j;$$

$$g_{m,n}^0 = 2\tau \varphi_{m,n}, \quad g_{m,n}^N = 2\tau \psi_{m,n}, \quad n = \overline{1, M-1}, m = \overline{1, M-1},$$

for the second order approximation.

In Tables 4-6, we give the results of the numerical calculations for both first and second order approximations in the cases $N = M = 10, 20, 40$. Table 1 presents the error between

Table 4 Error of v in example (57)

Difference scheme	$N = M = 10$	$N = M = 20$	$N = M = 40$
First order of the ADS	0.01089	0.00408	0.0017133
Second order of the ADS	5.39×10^{-3}	1.34×10^{-3}	3.27×10^{-4}

Table 5 Error of u in example (57)

Difference scheme	$N = M = 10$	$N = M = 20$	$N = M = 40$
First order of the ADS	0.026598	0.013488	6.77×10^{-3}
Second order of the ADS	7.47×10^{-4}	1.84×10^{-4}	3.95×10^{-5}

Table 6 Error of p in example (57)

Type of approximation	$N = M = 10$	$N = M = 20$	$N = M = 40$
First order of the ADS	0.41618	0.21802	0.11065
Second order of the ADS	2.71×10^{-3}	6.64×10^{-4}	1.40×10^{-4}

exact and approximately solutions of NBVP. Table 2 includes the error between exact and approximate u . Table 3 gives the error for p . It can be seen from Tables 1-3 that the second order of the ADS is more accurate compared with the first order of the ADS.

6 Conclusion

In the present paper, we discuss stability estimates for the solution of a Bitsadze-Samarskii type elliptic overdetermined multipoint NBVP. We apply the finite difference method to construct the first and second order of the ADSs for this problem and establish stability estimates for its solutions. The abstract results established are applied to get stability estimates for the solution of Bitsadze-Samarskii type overdetermined elliptic multidimensional differential and difference problems with multipoint NBVCs. Stability estimates for the solution of difference schemes are obtained. Finally, we present numerical results with explanation on the realization for two dimensional and three dimensional elliptic overdetermined multipoint NBVP in test examples.

Moreover, applying the results of [28, 32, 33] the high order ADSs for the numerical solution of a Bitsadze-Samarskii type elliptic overdetermined multipoint NBVP can be presented.

Competing interests

The author declares that he has no competing interests.

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