Note on small singular values of sequences of matrices

Charyyar Ashyralyyev and Zafer Cakir

Citation: AIP Conference Proceedings **1611**, 167 (2014); doi: 10.1063/1.4893824
View online: http://dx.doi.org/10.1063/1.4893824
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1611?ver=pdfcov
Published by the AIP Publishing

Note on small singular values of sequences of matrices

Charyyar Ashyralyyev*,† and Zafer Cakir*

*Department of Mathematical Engineering, Gumushane University, 29100, Gumushane, Turkey
†TAU, Ashgabat, Turkmenistan

Abstract. In this paper, some sequences of matrices for matrix pencils are considered. Properties of small singular values of these matrices are investigated. Estimates for singular values are obtained. The result of computer calculations is given.

Keywords: Matrix pencils, Singular values, Matrix theory
PACS: 02.10.Ud, 02.10.Yn, 02.60.Dc

INTRODUCTION

Matrix pencils play important role in numerical linear algebra, control systems and signal processing (see, for example [1–9] and the references therein). In [4], authors revalued the relation between the critical points of approximating the eigenvalues of matrix pencils and pseudospectra of perturbed pencils. Matrix pencils are also important tools in signal processing [6].

Let C be an $r \times s$ matrix. Decomposition C as with orthogonal matrices U, V and diagonal matrix K with nonnegative real diagonal elements is called the singular decomposition of matrix C [10]:

$$C = VKU^*. \tag{1}$$

The diagonal elements of the matrix K are called the singular values of matrix C and are denoted by

$$\sigma_1(C) = k_{11},\ \sigma_2(C) = k_{22}, \ldots, \sigma_r(C) = k_{rr}.$$

Singular values of the matrix C are nonnegative square roots of the eigenvalues of the symmetric matrices CC^* or C^*C. Usually, singular values are thought to be ordered, i.e.,

$$0 \le \sigma_1(C) \le \sigma_2(C) \le \ldots \le \sigma_r(C).$$

Let (A,B) be a pair of $m \times n$ matrices. For all $k = 0, 1, \ldots, m-1$, we consider the following sequence of matrices [1]

$$F_0 = \begin{bmatrix} A \\ B \end{bmatrix},\ F_1 = \begin{bmatrix} A & 0 \\ B & A \\ 0 & B \end{bmatrix},\ F_2 = \begin{bmatrix} A & 0 & 0 \\ B & A & 0 \\ 0 & B & A \\ 0 & 0 & B \end{bmatrix},$$

$$F_k = \begin{bmatrix} A & 0 & 0 & \cdots & 0 & 0 \\ B & A & 0 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\ 0 & 0 & 0 & \cdots & B & A \\ 0 & 0 & 0 & \cdots & 0 & B \end{bmatrix} \tag{2}$$

which play important roles in construction of Kronecker canonical form of matrix pencils. This canonical form provides many applications in control systems (see [2–9] and references therein).

In this work, we obtain estimates for singular values of sequences of matrices (2). We investigate the properties of small singular values of these matrices. Furthermore, results of computer calculations are given.

The rest of this paper is organized as follows. In Section 2, we present estimates for singular values of F_1. In Section 3, we obtain estimates for singular values of F_l. Section 4 is conclusion.

International Conference on Analysis and Applied Mathematics (ICAAM 2014)
AIP Conf. Proc. 1611, 167-171 (2014); doi: 10.1063/1.4893824

PROPERTY OF SMALL SINGULAR VALUES OF F_1

We investigate small singular values of F_0, F_1 and define relations between counter of small singular values of these matrices. From definition it follows that

$$F_0^* F_0 = [A^*A + B^*B],\ F_1^* F_1 = \begin{bmatrix} F_0^* F_0 & B^*A \\ A^*B & F_0^* F_0 \end{bmatrix}.$$

Denote that $N_0 = \min\{n, 2m\}$, $N_1 = \min\{2n, 3m\}$.

Theorem 1. *Assume that $\sigma_k(F_0),\ k = 1, \ldots, N_0$ and $\tau_i(F_1),\ i = 1, \ldots, N_1$ are the ordered singular values of F_0 and F_1, respectively. Let the following inequalities*

$$0 \le \sigma_1(F_0) \le \sigma_2(F_0) \le \cdots \le \sigma_{\rho_0}(F_0) \le \delta \tag{3}$$

hold for singular values of F_0, where $\delta \ge 0$ is a small number and ρ_0 is a natural number such that $\rho_0 \le N_0$. Then, for first $2\rho_0$ singular values of F_1 the inequalities

$$0 \le \tau_1(F_1) \le \tau_2(F_1) \le \cdots \le \tau_{\rho_0}(F_1) \le \cdots \le \tau_{2\rho_0}(F_1) \le \sqrt{2\rho_0}\,\delta \tag{4}$$

are satisfied.

Proof. Denote

$$H = F_1^* F_1 = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix},\ \overline{H} = \begin{bmatrix} H_{11} & 0_{m\times m} \\ 0_{m\times m} & H_{22} \end{bmatrix},$$

where $H_{11} = H_{22} = F_0^* F_0$, $H_{12} = B^*A$, $0_{m\times m}$ is $m \times m$ matrix with zero elements.

Let $\mu_1, \mu_2, \ldots, \mu_{2n}$ be the eigenvalues of the matrix H and $s_1, s_2, \ldots, s_{2n}$ be the eigenvalues of the matrix $\overline{H}$, here $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_{2n} \ge 0, s_1 \ge s_2 \ge \ldots \ge s_{2n} \ge 0$. By using K. Fan's theorem [11, 12], we get

$$\sum_{i=1}^{k} s_i \le \sum_{i=1}^{k} \mu_i \tag{5}$$

for each $1 \le k \le 2n$ and

$$\sum_{i=1}^{2n} s_i = \sum_{i=1}^{2n} \mu_i. \tag{6}$$

Subtracting (5) from (6) for fixed k, we obtain

$$\sum_{i=k+1}^{2n} s_i \ge \sum_{i=k+1}^{2n} \mu_i. \tag{7}$$

From (7) it follows that the inequalities

$$\begin{aligned} \mu_{2n} &\le s_{2n}, \\ \mu_{2n-1} &\le \mu_{2n-1} + \mu_{2n} \le s_{2n} + s_{2n-1}, \\ &\vdots \\ \mu_{2n-2\rho_0+1} &\le \sum_{i=2\rho_0+1}^{2n} \mu_{2n-i+1}. \end{aligned} \tag{8}$$

are valid.

First, suppose that $m \ge n$. By using definitions of eigenvalues and singular values, we have that

$$s_1 = \lambda_1,\ s_2 = \lambda_1,\ s_3 = \lambda_2,\ s_4 = \lambda_2, \ldots, s_{2n-1} = \lambda_n,\ s_{2n} = \lambda_n, \tag{9}$$

and

$$\lambda_1 = \sigma_n^2(F_0),\ \lambda_2 = \sigma_{n-1}^2(F_0), \ldots, \lambda_n = \sigma_1^2(F_0), \tag{10}$$

$$\mu_1 = \tau_{2n}^2(F_1),\ \mu_2 = \tau_{2n-1}^2(F_1), \ldots, \mu_n = \tau_{n+1}^2(F_1), \cdots, \mu_{2n} = \tau_1^2(F_1). \tag{11}$$

Applying (3), (7), and (9)-(11), we have

$$\tau_1^2(F_1) \le \sigma_1^2(F_0) \le \delta^2,$$

$$\tau_2^2(F_1) \le \sigma_1^2(F_0) + \sigma_1^2(F_0) \le 2\delta^2,$$

$$\tau_3^2(F_1) \le \sigma_1^2(F_0) + \sigma_1^2(F_0) + \sigma_2^2(F_0) \le 3\delta^2, \cdots,$$

$$\tau_{2\rho_0}^2(F_1) \le \sigma_1^2(F_0) + \sigma_1^2(F_0) + \sigma_2^2(F_0) + \sigma_2^2(F_0) + \cdots + \sigma_{\rho_0-1}^2(F_0) + \sigma_{\rho_0-1}^2(F_0) + \sigma_{\rho_0}^2(F_0) + \sigma_{\rho_0}^2(F_0) \le 2\rho_0\delta^2.$$

Hence, from these inequalities the required inequalities (4) follows for the case $m \ge n$.

Second, suppose that $m < n$. There are three subcases:

$$2m < n; \tag{12}$$

$$2m \ge n \text{ and } 3m < n; \tag{13}$$

$$2m \ge n \text{ and } 3m \ge n. \tag{14}$$

Let us consider subcase (12). We have $3m < n$. From definition of singular values it follows that formula (9) is valid. Hence, we get

$$\lambda_n = \lambda_{n-1} = \ldots = \lambda_{2m-1} = 0,$$

$$\lambda_{2m} = \sigma_1^2(F_0),\ \lambda_{2m+1} = \sigma_2^2(F_0), \ldots, \lambda_1 = \sigma_{2m}^2(F_0), \tag{15}$$

$$\mu_{2n} = \mu_{2n-1} = \cdots = \mu_{3m+1} = 0,\ \mu_{3m} = \tau_1^2(F_1), \tag{16}$$

$$\mu_{3m-1} = \tau_2^2(F_1), \ldots, \mu_1 = \tau_{3m}^2(F_1).$$

Applying (7) for $k = 2n-1, 2n-2, \cdots, 3m-2\rho_0-1$ and inequalities (9), (15), (16), we have that inequalities (4) are valid.

Now, we consider subcase (13). By using definition of singular values, we have (9), (10), (15). Applying (7) for $k = 2n-1, 2n-2, \cdots, 3m-2\rho_0-1$ and inequalities (3), (9), (10), (15), we get (4).

Finally, we consider subcase (14). By using definition of singular values, we obtain (9)-(11). In a similar manner as case $m \ge n$, we can get (4). The proof of Theorem 1 is completed. □

PROPERTY OF SMALL SINGULAR VALUES OF F_l

Consider F_l ,$1 < l < m$. Denote $N_l = \min\{(l+1)n, (l+2)m\}$. It is easy to get

$$F_l^* F_l = \begin{bmatrix} F_{l-1}^* F_{l-1} & Q_l \\ Q_l^* & F_0^* F_0 \end{bmatrix},\ Q_l = \begin{bmatrix} 0_{nl\times n} \\ B^* A \end{bmatrix},$$

where $0_{nl\times n}$ is $nl \times n$ matrix with zero elements.

Theorem 2. *Suppose that $\sigma_i(F_{l-1})$ $(i = 1, \ldots, N_{l-1})$, $\eta_k(F_0)$ $(k = 1, \ldots, N_0)$, and $\tau_j(F_l)$ $(j = 1, \ldots, N_l)$ are the ordered singular values of F_{l-1} and F_l, respectively. Let the following inequalities*

$$\begin{aligned} &0 \le \sigma_1(F_{l-1}) \le \sigma_2(F_{l-1}) \le \cdots \le \sigma_{\rho_{l-1}}(F_{l-1}) \le \delta, \\ &0 \le \eta_1(F_0) \le \eta_2(F_0) \le \cdots \le \eta_{\rho_0}(F_0) \le \delta \end{aligned} \tag{17}$$

be satisfied for singular values of F_{l-1} and F_0 where $\delta \ge 0$ is a small number and ρ_{l-1} is a natural number such that $\rho_{l-1} \le N_{l-1}$. Then, $(p_{l-1} + \rho_0)$ first singular values of F_l are small and the inequalities

$$0 \le \tau_1(F_l) \le \tau_2(F_l) \le \cdots \le \tau_{\rho_{l-1}}(F_l) \le \cdots \le \tau_{\rho_{l-1}+\rho_0}(F_l) \le \sqrt{\rho_{l-1}+\rho_0}\,\delta$$

are valid.

Consider F_k, $2 < k < m$. Denote $N_k = \min\{(k+1)n,\ (k+2)m\}$. Let $0 \le l,\ q < m$, such that $l+q < m$. We can obtain

$$F^*_{l+q+1}F_{l+q+1} = \begin{bmatrix} F^*_l F_l & P_{lq} \\ P^*_{lq} & F^*_q F_q \end{bmatrix},\ P_{lq} = \begin{bmatrix} 0_{n(l+1)\times n} & 0_{n(l+1)\times n(q+1)} \\ B^*A & 0_{n\times n(q+1)} \end{bmatrix},$$

where $0_{n(l+1)\times n(q+1)}$ is $n(l+1)\times n(q+1)$ matrix with zero elements.

Theorem 3. *Assume that $\sigma_i(F_l)$ $(i = 1,\dots,N_l)$, $\eta_k(F_q)$ $(k = 1,\dots,N_q)$, and $\tau_j(F_{l+q+1})$, $\left(j = 1,\dots,N_{l+q+1}\right)$ are the ordered singular values of F_l, F_q and F_l, respectively. Let the following inequalities*

$$\begin{aligned} &0 \le \sigma_1(F_l) \le \sigma_2(F_l) \le \cdots \le \sigma_{\rho_l}(F_l) \le \delta, \\ &0 \le \eta_1(F_q) \le \eta_2(F_q) \le \cdots \le \eta_{\rho_q}(F_q) \le \delta \end{aligned} \tag{18}$$

be satisfied for singular values of F_l and F_q, where $\delta \ge 0$ is a small number and ρ_l is a natural number such that $\rho_l \le N_l$. Then $(p_l + \rho_q)$ first singular values of F_{l+q+1} are small and the inequalities

$$0 \le \tau_1(F_{l+q+1}) \le \tau_2(F_{l+q+1}) \le \cdots \le \tau_{\rho_l}(F_{l+q+1}) \le \cdots \le \tau_{\rho_l+\rho_q}(F_{l+q+1}) \le \sqrt{\rho_l+\rho_q}\,\delta$$

are satisfied.

Now, we shall consider the following example:

$$n = 4,\ m = 5,\ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Matlab software is used to calculate the singular values of sequence matrices. Table 1 presents singular values of $F_0,\ F_1,\ F_2,\ F_3$.

TABLE 1. Singular values of matrices

Singular values of F_0	0	1	1	1.4142	1.4142
Singular values of F_1	0	0	0.61803	1	1
	1	1	1.4142	1.618	1.7321
Singular values of F_2	0	0	0	0.44504	0.61803
	1	1	1	1	1
	1.247	1.4142	1.618	1.7321	1.8019
Singular values of F_3	0	0	0	1.282e-016	0.44504
	0.44504	0.61803	1	1	1
	1	1	1	1.247	1.247
	1.4142	1.618	1.7321	1.8019	1.8019

For small number $\delta = 1.3\cdot 10^{-16}$ we have $\rho_0 = 1,\ \rho_1 = 2,\ \rho_2 = 3,\ \rho_3 = 4$. As it can be seen from Table 1, numbers of small singular values for matrices $F_0,\ F_1,\ F_2,\ F_3$ are $1, 2, 3, 4$, respectively.

CONCLUSION

We consider some sequences of matrices for matrix pencils. Properties of small singular values of these matrices are investigated. Estimates for singular values are obtained. Example with computer calculations is given.

REFERENCES

1. F. R. Gantmacher, *The Theory of Matrices 1, 2*, American Mathematical Society Chelsea, New York, 1959.
2. L. Dai, *Singular Control Systems*, Springer Verlag, New York, 1989.
3. A. Diaz, and M. I. Garcia-Planas, *14th WSEAS International Conference on Applied Mathematics*, Puerto de la Cruz, Spain, edited by C.A.Bulucea, V. Mladenov, E. Pop, et al. (Recent Advances in Applied Mathematics, Book Series: Math. Comput. Sci. Eng.) (WSEAS, Athens, Greece, 2009) 64-69.
4. Sk. S. Ahmad, R. Alam, and R. Byers, *SIAM J. Matrix Anal. Appl.* **31**, 1915-1933 (2010).
5. R. C. Li, *Math.Comp.* **72**, 715-728 (2003).
6. C. Chang, Z. Ding, S. F. Yau, and F. H. Y. Chan, *IEEE Trans. Automat. Contr.* **48**, 900-907 (2000).
7. S. Wang, X. Guan, D. Wang, X. Ma, and Y. Su, *Electronic Letters* **43**, 3-5 (2007).
8. F. R. Chang, and H. C. Chen, *Syst. Control Lett.* **18**, 179-182 (1992).
9. A. S. Morse, *SIAM J. Contr.* **11**, 446-465, (1973).
10. A. G. Antonov, S. K. Godunov, O. P. Kirriluk, and V. I. Costin, *Guaranteed Accuracy of Solution of Linear Equations System in Euclide Space*, Nauka, Novosibirsk, 1988.
11. K. Fan, Inequalities for eigenvalues of Hermitian matrices, Contributions to the solution of systems of linear equations and the determination of eigenvalues, National Bureau of Standards Applied Mathematics Series, U. S. Government Printing Office, Washington, D.C., **no. 39** 131-139 (1954).
12. A. W. Marshall, and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.