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High Order Approximation of the Inverse Elliptic Problem with Dirichlet-Neumann Conditions

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Abstract. Inverse problem for the multidimensional elliptic equation with Dirichlet-Neumann conditions is considered. High order of accuracy difference schemes for the solution of inverse problem are presented. Stability, almost coercive stability and coercive stability estimates of the third and fourth orders of accuracy difference schemes for this problem are obtained. Numerical results in a two dimensional case are given.

1. Introduction

Methods of solution of the inverse problems for partial differential equations have been investigated extensively by many researchers (see [1]-[18] and the references therein).

Consider inverse problem of finding functions $u(t,x)$ and $p(x)$ for the multidimensional elliptic equation with the following boundary conditions

$$\begin{cases} -u_{tt}(t,x) - \sum\limits_{q=1}^{n} (a_q(x)u_{x_q})_{x_q} + \sigma u(t,x) = f(t,x) + p(x), \\ x = (x_1, \cdots, x_n) \in \Omega,\ 0 < t < T, \\ u(0,x) = \varphi(x), u(T,x) = \psi(x), u(\lambda,x) = \xi(x), x \in \overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial \vec{n}} = 0,\ x \in S, 0 \le t \le T. \end{cases} \tag{1.1}$$

Here, $0 < \lambda < T$ and $\sigma > 0$ are given numbers, $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,T),\ x \in \Omega)$ are given smooth functions and $a_r(x) \ge a > 0$ $(x \in \Omega)$, and $\Omega = (0,\ell) \times \cdots \times (0,\ell)$ is the open cube in the n-dimensional Euclidean space with boundary S, $\overline{\Omega} = \Omega \cup S$.

Well-posedness and .the first and second order of accuracy in t and the second order of accuracy in space variables for the approximate solution of problem (1.1) was investigated in [12]. High order of accuracy stable difference schemes for nonlocal boundary value elliptic problems presented in [19]-[21].

For the differential operator A^x generated by problem (1.1),

$$A^x u = -\sum_{q=1}^{n} (a_q(x)u_{x_q})_{x_q} + \sigma u, \tag{1.2}$$

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it follows that (see [27], [28]) $B = \frac{1}{2}(\tau C + \sqrt{4C + \tau^2 C^2})$ is a self-adjoint positive definite operator and $R = (I + \tau B)^{-1}$ which is defined on the whole space H is a bounded operator. Here, $C = A^x + \frac{\tau^2}{12}(A^x)^2$ and I is the identity operator.

Now we give some lemmas that will be needed below.

Lemma 1.1. *([13]). For $1 \le l \le N-1$, the operator*

$$S_1 = I - R^{2N} - (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)$$
$$\times(R^{l-1} + R^{2N-l+1} - R^{N-l+1} + R^{N+l-1}) - (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)(R^l + R^{2N-l} - R^{N-l} + R^{N+l})$$
$$-(-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)(R^{l+1} + R^{2N-l-1} - R^{N-l-1} + R^{N+l+1})$$

has an inverse such that $G_1 = S_1^{-1}$, and estimate

$$\| G_1 \|_{H\to H} \le M(\delta). \tag{1.3}$$

is valid.

Lemma 1.2. *([13]). For $1 \le l \le N-1$ the operator*

$$S_2 = I - R^{2N} - (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)(R^{l-2} + R^{2N-l+2} - R^{N-l+2} + R^{N+l-2})$$
$$-(-\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 + \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)(R^{l-1} + R^{2N-l+1} - R^{N-l+1} + R^{N+l-1})$$
$$+(\frac{\lambda}{\tau} - l)^2(R^l + R^{2N-l} - R^{N-l} + R^{N+l}) - (\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 - \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)$$
$$\times(R^{l+1} + R^{2N-l-1} - R^{N-l-1} + R^{N+l+1}) - (-\frac{1}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)$$
$$\times(R^{l+2} + R^{2N-l-2} - R^{N-l-2} + R^{N+l+2}).$$

has an inverse such that $G_2 = S_2^{-1}$, and the estimate

$$\| G_2 \|_{H\to H} \le M \tag{1.4}$$

is satisfied.

Our aim in this paper is construction of high order accuracy stable difference schemes for inverse problem (1.1). In this work, we present the third and fourth orders of accuracy in t and the second order of accuracy in space variables for the approximate solution of problem (1.1). Stability, almost coercive stability and coercive stability estimates of these difference schemes are obtained. The modified Gauss elimination method is applied for testing the third and fourth orders of accuracy difference schemes in a two dimensional case.

The remainder of this paper is organized as follows. In Section 2, we present the third and fourth order difference schemes for problem (1.1) and establish their well-posedness. In Section 3, we give the numerical results. Section 4 is conclusion.

2. The third and fourth orders accuracy of difference schemes and the stability estimates

The discretization of problem (1.1) is carried out in two steps. In the first step, we define the grid spaces

$$\begin{aligned}\widetilde{\Omega}_h &= \{x_m = (h_1m_1, \cdots, h_nm_n); m = (m_1, \cdots, m_n), m_q = 0, \cdots, M_q,\\ h_qM_q &= \ell, q = 1, \cdots, n\}, \Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.\end{aligned}$$

Introduce the Hilbert spaces $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ of grid functions $\rho^h(x) = \{\rho(h_1m_1, \cdots, h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\left\|\rho^h\right\|_{L_{2h}} = (\sum_{x\in\widetilde{\Omega}_h} |\rho^h(x)|^2 h_1 \cdots h_n)^{1/2},$$

$$\left\|\rho^h\right\|_{W_{2h}^2} = \left\|\rho^h\right\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{q=1}^{n} \left|(\rho^h)_{x_q}\right|^2 h_1 \cdots h_n\right)^{1/2}$$

$$+ \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{q=1}^{n} \left|(\rho^h(x))_{x_q\overline{x_q},\, m_q}\right|^2 h_1 \cdots h_n\right)^{1/2}.$$

To the differential operator A^x (1.2), we assign the difference operator A_h^x defined by the formula,

$$A_h^x u^h(x) = -\sum_{q=1}^{n} \left(a_q(x) u_{\overline{x}_q}^h\right)_{x_q, j_r} + \sigma u^h(x) \tag{2.1}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $D^h u^h(x) = 0$ for all $x \in S_h$. Here $D^h u^h(x)$ is an approximation of $\frac{\partial u}{\partial \vec{n}}$.
By using A_h^x, for obtaining $u^h(t,x)$ functions we arrive at problem

$$\begin{cases}-\frac{d^2u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x) + p^h(x),\ 0 < t < T,\ x \in \Omega_h,\\ u^h(0,x) = \varphi^h(x),\ u^h(\lambda,x) = \xi^h(x), u^h(T,x) = \psi^h(x), x \in \widetilde{\Omega}_h.\end{cases} \tag{2.2}$$

For finding a solution $u^h(t,x)$ of the problem (2.2), we apply the substitution

$$u^h(t,x) = v^h(t,x) + (A_h^x)^{-1} p^h(x). \tag{2.3}$$

Here $v^h(t,x)$ is the solution of the following nonlocal boundary value problem

$$\begin{cases}-\frac{d^2v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x),\ 0 < t < T,\ x \in \Omega_h,\\ v^h(0,x) - v^h(\lambda,x) = \varphi^h(x) - \xi^h(x),\ v^h(T,x) - v^h(\lambda,x) = \psi^h(x) - \xi^h(x), x \in \widetilde{\Omega}_h\end{cases} \tag{2.4}$$

for a system of ordinary differential equations, and $p^h(x)$ is unknown function which is defined by formula

$$p^h(x) = A_h^x \varphi^h(x) - A_h^x v^h(0,x), x \in \widetilde{\Omega}_h. \tag{2.5}$$

We consider the algorithm for solving the problem (2.2) which includes three stages ([11]). In the first stage, consider the nonlocal boundary value problem (2.4) and obtain $v^h(t,x)$. In the second stage, putting $t = 0$, find $v^h(0,x)$. Then, applying (2.5), obtain $p^h(x)$. In the third stage, use formula (2.3) for obtaining the solution $u^h(t,x)$ of problem (2.2).

In the second step, we approximate (2.2) in variable t. Let $[0,T]_\tau = \{t_k = k\tau, k = 1,\cdots,N,\ N\tau = T\}$ be the uniform grid space with step size $\tau > 0$, where N is a fixed positive integer.

To formulate our result on well-posedness of difference schemes, we give definition of $C([0,T]_\tau, H)$ and $C_{0T}^{\alpha,\alpha}([0,T]_\tau, H)$ which are the lineer spaces of mesh functions $\theta^\tau = \{\theta_k\}_{k=1}^{N-1}$ with values in the Hilbert space H. We denote $C([0,T]_\tau, H)$ normed space with the norm

$$\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C([0,T]_\tau,H)} = \max_{1\le k\le N-1} \|\theta_k\|_H\,,$$

and $C_{0T}^{\alpha,\alpha}([0,T]_\tau, H)$ normed space with the norm

$$\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}([0,T]_\tau,H)} = \left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C([0,T]_\tau,H)}$$

$$+\sup_{1\le k<k+n\le N-1} \frac{(k\tau+n\tau)^\alpha (T-k\tau)^\alpha \|\theta_{k+n}-\theta_k\|_H}{(n\tau)^\alpha}.$$

Applying the approximate formula

$$u^h(\lambda) = (\frac{1}{2}(\frac{\lambda}{\tau}-l) + \frac{1}{2}(\frac{\lambda}{\tau}-[\frac{\lambda}{\tau}])^2)u^h((l-1)\tau)$$

$$+(1-(\frac{\lambda}{\tau}-l)^2)u^h(l\tau) + (-\frac{1}{2}(\frac{\lambda}{\tau}-l+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)u^h((l+1)\tau) + o(\tau^3),$$

for $u^h(\lambda) = \xi^h$, the problem (2.2) is replaced by the third order of accuracy difference scheme

$$\begin{aligned}
&-\tau^{-2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + A_h^x u_k^h(x) + \frac{\tau^2}{12}(A_h^x)^2 u_k^h(x) = \theta_k^h(x) + p^h(x),\\
&\theta_k^h(x) = f^h(t_k,x) + \frac{\tau^2}{12}\left(\frac{f^h(t_{k+1},x) - 2f^h(t_k,x) + f^h(t_{k-1},x)}{\tau^2} + A_h^x f^h(t_k,x)\right),\\
&t_k = k\tau,\ 1\le k\le N-1,\ x\in\Omega_h, u_0^h(x) = \varphi^h(x),\ u_N^h(x) = \psi^h(x), x\in\widetilde{\Omega}_h,\\
&(\frac{1}{2}(\frac{\lambda}{\tau}-l) + \frac{1}{2}(\frac{\lambda}{\tau}-l)^2)u_{l-1}^h(x) + (1-\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)u_l^h(x)\\
&+(-\frac{1}{2}(\frac{\lambda}{\tau}-l) + \frac{1}{2}(\frac{\lambda}{\tau}-l)^2)u_{l+1}^h(x) = \xi^h(x),\ x\in\widetilde{\Omega}_h
\end{aligned}\tag{2.6}$$

Here $l = \left[\frac{\lambda}{\tau}\right]$, $[\cdot]$ is a notation for greatest integer function.

By using the approximate formula

$$u^h(\lambda) = (-\frac{1}{12}(\frac{\lambda}{\tau}-l) + \frac{1}{12}(\frac{\lambda}{\tau}-l)^3)u^h((l-2)\tau) + (\frac{8}{12}(\frac{\lambda}{\tau}-l) + \frac{1}{2}(\frac{\lambda}{\tau}-l)^2$$

$$-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)u^h((l-1)\tau) + (1-(\frac{\lambda}{\tau}-l)^2)u^h(l\tau) + (-\frac{8}{12}(\frac{\lambda}{\tau}-l) + \frac{1}{2}(\frac{\lambda}{\tau}-l)^2$$

$$+\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)u^h((l+1)\tau) + (\frac{1}{12}(\frac{\lambda}{\tau}-l) - \frac{1}{12}(\frac{\lambda}{\tau}-l)^3)u^h((l+2)\tau) + o(\tau^4)$$

for $u^h(\lambda) = \xi^h$, the problem (2.2) is replaced by the fourth order of accuracy difference scheme

$$-\tau^{-2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + A_h^x u_k^h(x) + \frac{\tau^2}{12}(A_h^x)^2 u_k^h(x) = \theta_k^h(x) + p^h(x),$$

$$\theta_k^h(x) = f^h(t_k,x) + \frac{\tau^2}{12}\left(\frac{f^h(t_{k+1},x) - 2f^h(t_k,x) + f^h(t_{k-1},x)}{\tau^2} + A_h^x f^h(t_k,x)\right),$$

$$t_k = k\tau,\ 1 \le k \le N-1,\ x \in \Omega_h, u_0^h(x) = \varphi^h(x),\ u_N^h(x) = \psi^h(x), x \in \widetilde{\Omega}_h,$$
$$(\frac{1}{12}(\frac{\lambda}{\tau}-l) - \frac{1}{12}(\frac{\lambda}{\tau}-l)^3)u_{l-2}^h(x) + (-\frac{8}{12}(\frac{\lambda}{\tau}-l) + \frac{1}{2}(\frac{\lambda}{\tau}-l)^2 + \frac{1}{6}(\frac{\lambda}{\tau}-l)^3)u_{l-1}^h(x)$$
$$+(1-(\frac{\lambda}{\tau}-l)^2)u_l^h(x) + (\frac{8}{12}(\frac{\lambda}{\tau}-l) + \frac{1}{2}(\frac{\lambda}{\tau}-l)^2 - \frac{1}{6}(\frac{\lambda}{\tau}-l)^3)u_{l+1}^h(x)$$
$$+(-\frac{1}{12}(\frac{\lambda}{\tau}-l) + \frac{1}{12}(\frac{\lambda}{\tau}-l)^3)u_{l+2}^h(x) = \xi^h(x),\ x \in \widetilde{\Omega}_h. \tag{2.7}$$

Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers.

Theorem 2.1. *The solutions* $\left(\left\{u_k^h\right\}_{k=1}^{N-1}, p^h\right)$ *of difference schemes (2.6) and (2.7) obey the following stability estimates:*

$$\left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C([0,T]_\tau,L_{2h})} \le M_1(\delta)\left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \left\|\xi^h\right\|_{L_{2h}} + \left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C([0,T]_\tau,L_{2h})}\right],$$

$$\left\|p^h\right\|_{L_{2h}} \le M_1(\delta)\left[\left\|A\varphi^h\right\|_{W_{2h}^2} + \left\|A\psi^h\right\|_{W_{2h}^2} + \left\|A\xi^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}([0,T]_\tau,L_{2h})}\right],$$

where $M_1(\delta)$ *does not depend on* $\tau, \alpha, \varphi^h, \psi^h, \xi^h$, *and* $\left\{f_k^h\right\}_{k=1}^{N-1}, 1 \le k \le N-1.$

Theorem 2.2. *The solutions of difference schemes (2.6) and (2.7) obey the following almost coercive stability estimate:*

$$\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2})\right\}_1^{N-1}\right\|_{C([0,T]_\tau,L_{2h})} + \left\|\left\{\left(A + \frac{\tau^2}{12}A^2\right)u_k\right\}_{k=1}^{N-1}\right\|_{C([0,T]_\tau,W_{2h}^2)}$$
$$+\left\|p^h\right\|_{L_{2h}} \le M_2(\delta)\left[\ln\left(\frac{1}{\tau+h}\right)\left\|\left\{f_k^h\right\}_1^N\right\|_{C([0,T]_\tau,L_{2h})} + \left\|\left(A + \frac{\tau^2}{12}A^2\right)\varphi^h\right\|_{W_{2h}^2}\right.$$
$$\left. + \left\|\left(A + \frac{\tau^2}{12}A^2\right)\psi^h\right\|_{W_{2h}^2} + \left\|\left(A + \frac{\tau^2}{12}A^2\right)\xi^h\right\|_{W_{2h}^2}\right].$$

Here, $M_2(\delta)$ *is independent of* $\tau, \alpha, \varphi^h, \psi^h, \xi^h$, *and* $\left\{f_k^h\right\}_{k=1}^{N-1}, 1 \le k \le N-1.$

Theorem 2.3. *The solutions of difference schemes (2.6) and (2.7) obey the following coercive stability estimate:*

$$\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2})\right\}_1^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}([0,T]_\tau,L_{2h})} + \left\|\{u_k\}_{k=1}^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}([0,T]_\tau,W_{2h}^2)} + \left\|p^h\right\|_{L_{2h}}$$
$$\le M_3(\delta)\left[\frac{1}{\alpha(1-\alpha)}\left\|\left\{f_k^h\right\}_1^N\right\|_{C_{0T}^{\alpha,\alpha}([0,T]_\tau,L_{2h})} + \left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^2} + \left\|\xi^h\right\|_{W_{2h}^2}\right],$$

where $M_3(\delta)$ *does not depend on* $\tau, \alpha, \varphi^h, \psi^h, \xi^h$, *and* $\left\{f_k^h\right\}_{k=1}^{N-1}, 1 \le k \le N-1.$

Proofs of Theorems 2.1 - 2.3 are based on the symmetry property of operator A^x defined by (2.1) and on the following formulas:

$$u_k^h(x) = (I - R^{2N})^{-1}[((R^k - R^{2N-k}) v_0^h(x) + (R^{N-k} - R^{N+k}) v_N^h(x))$$

$$-(R^{N-k} - R^{N+k})(I + \tau B)(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i} - R^{N+i})\theta_i^h(x)\tau]$$

$$+(I + \tau B)(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{|k-i|} - R^{k+i})\theta_i^h(x)\tau + \varphi^h(x) - v_0^h(x),$$

$$p^h(x) = A_h^x\varphi^h(x) - A_h^x v_0^h(x), v_N^h(x) = v_0^h(x) + \psi^h(x) - \varphi^h(x),$$

$$v_0^h(x) = (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)G_1(R^{N-l+1} - R^{N+l-1})(I + \tau B)$$

$$\times(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i} - R^{N+i})\theta_i^h(x)\tau + G_1(I - R^{2N})(I + \tau B)$$

$$\times(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{|l-1-i|} - R^{l-1+i})\theta_i^h(x)\tau + (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)G_1$$

$$\times(R^{N-l-1} - R^{N+l+1})(I + \tau B)(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i} - R^{N+i})\theta_i^h(x)\tau$$

$$+G_1(I - R^{2N})(I + \tau B)(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{|l+1-i|} - R^{l+1+i})\theta_i^h(x)\tau$$

$$+(-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)G_1(R^{N-l-1} - R^{N+l+1})(I + \tau B)$$

$$\times(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i} - R^{N+i})\theta_i^h(x)\tau + G_1(I - R^{2N})(I + \tau B)$$

$$\times(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{|l+1-i|} - R^{l+1+i})\theta_i^h(x)\tau + G_1(I - R^{2N})(\varphi^h(x) - \xi^h(x))$$

$$+G_1((\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)(R^{N-l+1} - R^{N+l-1}) + (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)$$

$$\times(R^{N-l-1} - R^{N+l+1}) + (-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)(R^{N-l-1} - R^{N+l+1}))(\psi^h(x) - \varphi^h(x)),$$

for difference scheme (2.6),

$$v_0^h(x) = (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)G_2(R^{N-l+2} - R^{N+l-2})(I + \tau B)$$

$$\times(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i} - R^{N+i})\theta_i^h(x)\tau + G_2(I - R^{2N})(I + \tau B)$$

$$\times(2I + \tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{|l-2-i|} - R^{l-2+i})\theta_i^h(x)\tau$$

$$+(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2+\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)G_2\left(R^{N-l+1}-R^{N+l-1}\right)$$
$$\times(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i^h(x)\tau+G_2(I-R^{2N})(I+\tau B)$$
$$\times(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|l-1-i|}-R^{l-1+i}\right)\theta_i^h(x)\tau+(1-(\frac{\lambda}{\tau}-l)^2)G_2$$
$$\times\left(R^{N-l}-R^{N+l}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i^h(x)\tau$$
$$+G_2(I-R^{2N})(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|l-i|}-R^{l+i}\right)\theta_i^h(x)\tau$$
$$+(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)G_2\left(R^{N-l-1}-R^{N+l+1}\right)$$
$$\times(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i^h(x)\tau+G_2(I-R^{2N})(I+\tau B)$$
$$\times(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|l+1-i|}-R^{l+1+i}\right)\theta_i^h(x)\tau+(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)$$
$$\times G_2\left(R^{N-l-2}-R^{N+l+2}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i^h(x)\tau$$
$$+G_2(I-R^{2N})(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|l+2-i|}-R^{l+2+i}\right)\theta_i^h(x)\tau$$
$$+G_2(I-R^{2N})\left(\varphi^h(x)-\xi^h(x)\right)+G_2((\frac{1}{12}(\frac{\lambda}{\tau}-l)-\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)\left(R^{N-l-2}-R^{N+l+2}\right)$$
$$+(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2+\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)\left(R^{N-l+1}-R^{N+l-1}\right)$$
$$+(1-(\frac{\lambda}{\tau}-l)^2)\left(R^{N-l}-R^{N+l}\right)+(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)$$
$$\times\left(R^{N-l-1}-R^{N+l+1}\right)+(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)\left(R^{N-l-2}-R^{N+l+2}\right))(\psi^h(x)-\varphi^h(x)),$$

for difference scheme (2.7) and on the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 2.4. *([29]). For the solution of the elliptic difference problem*

$$\begin{cases}A_h^x u^h(x)=\omega^h(x),\ x\in\widetilde{\Omega}_h,\\ D^h u^h(x)=0,\ x\in S_h,\end{cases}$$

the following coercivity inequality holds :

$$\sum_{q=1}^{n}\left\|(u_k^h)_{\bar{x}_q\bar{x}_q,j_q}\right\|_{L_{2h}}\leq M\|\omega^h\|_{L_{2h}},$$

where M does not depend on h and ω^h.

3. Numerical Results

For the numerical result, we consider the inverse problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{\partial u(t,x)}{\partial x}\right) + u(t,x) = f(t,x) + p(x), \\ f(t,x) = (\exp(-t) + 2t)\cos(x), 0 < x < \pi, 0 < t < T, \\ u(0,x) = 2\cos(x), u(T,x) = (\exp(-T) + T + 1)\cos(x), \\ u(\lambda,x) = (\exp(-\lambda) + \lambda + 1)\cos(x), 0 \le x \le \pi, \\ u_x(t,0) = u_x(t,\pi) = 0, \ 0 \le t \le T \quad (T = 1, \lambda = \frac{4}{7}T). \end{cases} \tag{3.1}$$

for the two dimensional elliptic equation. It is clear that $u(t,x) = (\exp(-t) + t + 1)\cos(x)$ and $p(x) = 2\cos(x)$ are the exact solutions of (3.1).

We represent $u(t,x)$ by formula $u(t,x) = v(t,x) + w(t,x)$, where $v(t,x)$ is the solution of the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2 v(t,x)}{dt^2} - \frac{\partial}{\partial x}\left(\frac{\partial v(t,x)}{\partial x}\right) + v(t,x) = f(t,x), 0 < x < \pi, 0 < t < T, \\ v(0,x) - v(\lambda,x) = (1 - \exp(-\lambda) - \lambda)\cos(x), 0 \le x \le \pi, \\ v(T,x) - v(\lambda,x) = (\exp(-T) - \exp(-\lambda) + T - \lambda)\cos(x), 0 \le x \le \pi, \\ v_x(t,0) = v_x(t,\pi) = 0, 0 \le t \le T, \end{cases} \tag{3.2}$$

and $w(t,x)$ is the solution of the boundary value problem

$$\begin{cases} -\frac{d^2 w(t,x)}{dt^2} - \frac{\partial}{\partial x}\left(\frac{\partial w(t,x)}{\partial x}\right) + w(t,x) = p(x), \ 0 < x < \pi, 0 < t < T, \\ w(0,x) = (\exp(-\lambda) + \lambda + 1)\cos(x) - v(\lambda,x), \ 0 \le x \le \pi, \\ w(T,x) = (\exp(-\lambda) + \lambda + 1)\cos(x) - v(\lambda,x), \ 0 \le x \le \pi, \\ w_x(t,0) = w_x(t,\pi) = 0, 0 \le t \le T. \end{cases} \tag{3.3}$$

Introduce small parameters τ and h such that $N\tau = T, Mh = \pi$. For approximate solution of nonlocal boundary value problem (3.2), consider the set $[0,T]_\tau \times [0,\pi]_h$of a family of grid points

$$[0,T]_\tau \times [0,\pi]_h = \{(t_k, x_n) : t_k = k\tau, \ k = 1, \cdots, N-1, x_n = nh, \ n = 1, \cdots, M-1\}.$$

Applying (3.4) and the following formulas for approximation of sufficiently smooth function ρ :

$$\frac{\rho(x_{n+1}) - \rho(x_{n-1})}{2h} - \rho'(x_n) = O(h^2), \frac{\rho(x_{n+1}) - 2\rho(x_n) + \rho(x_{n-1})}{h^2} - \rho''(x_n) = O(h^2), \tag{3.4}$$

$$\frac{10\rho(0) - 15\rho(h) + 6\rho(2h) - \rho(3h)}{h^3} - \rho'''(0) = O(h^2),$$

$$\frac{-3\rho(0) + 4\rho(h) - \rho(2h)}{2h} - \rho'(0) = O(h^2),$$

$$\frac{10\rho(\pi) - 15\rho(\pi - h) + 6\rho(\pi - 2h) - \rho(\pi - 3h)}{h^3} - \rho'''(\pi) = O(h^2),$$

$$\frac{-3\rho(\pi) + 4\rho(\pi - h) - \rho(\pi - 2h)}{2h} - \rho'(\pi) = O(h^2),$$

we get, respectively,

$$-\frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} - \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + v_n^k \tag{3.5}$$

$$-\frac{\tau^2}{12}\left[-\frac{1}{h^2}\left(-\frac{v_{n+2}^k - 2v_{n+1}^k + v_n^k}{h^2} + v_{n+1}^k\right) + \frac{2}{h^2}\left(-\frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + v_n^k\right)\right.$$

$$-\frac{1}{h^2}\left(\frac{v_n^k-2v_{n-1}^k+v_{n-2}^k}{h^2}+v_{n-1}^k\right)-\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2}+v_n^k\Bigg]$$
$$=(\exp(-t_k)+2t_k)\cos(x_n)+\frac{\tau^2}{12}(\exp(-t_k)+4t_k)\cos(x_n)),$$
$$k=1,\cdots,N-1,\ n=2,\cdots,M-2,$$
$$-3v_0^k+4v_1^k-v_2^k=0,-3v_M^k+4v_{M-1}^k-v_{M-2}^k=0,$$
$$10v_0^k-15v_1^k+6v_2^k-v_3^k=10v_M^k-15v_{M-1}^k+6v_{M-2}^k-v_{M-3}^k=0,k=0,\cdots,N,$$
$$v_n^0-(\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l-1}-(1-\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^l$$
$$-(-\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l+1}=(1-\exp(-\lambda)-\lambda)\cos(x_n),n=0,\cdots,M,$$
$$v_n^N-(\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l-1}-(1-\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^l-(-\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l+1}$$
$$=(\exp(-t_N)-\exp(-\lambda)+t_N-\lambda)\cos(x_n),n=0,\cdots,M,$$

the third order of accuracy in t and second order accuracy in x for the approximate solution of the nonlocal boundary value problem (3.2), and

$$-\frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k \tag{3.6}$$
$$-\frac{\tau^2}{12}\left[-\frac{1}{h^2}\left(-\frac{w_{n+2}^k-2w_{n+1}^k+w_n^k}{h^2}+w_{n+1}^k\right)+\frac{2}{h^2}\left(-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k\right)\right.$$
$$\left.-\frac{1}{h^2}\left(\frac{w_n^k-2w_{n-1}^k+w_{n-2}^k}{h^2}+w_{n-1}^k\right)-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k\right]$$
$$=p(x_n)+\frac{\tau^2}{6}p(x_n),k=1,\cdots,N-1,\ n=2,\cdots,M-2,$$
$$-3w_0^k+4w_1^k-w_2^k=-3w_M^k+4w_{M-1}^k-w_{M-2}^k=0,$$
$$10w_0^k-15w_1^k+6w_2^k-w_3^k=10w_M^k-15w_{M-1}^k+6w_{M-2}^k-w_{M-3}^k=0,k=0,\cdots,N,$$
$$w_n^0=(\exp(-\lambda)+\lambda+1)\cos(x_n)-(\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l-1}$$
$$-(1-\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^l-(-\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l+1},n=0,\cdots,M,$$
$$w_n^N=(\exp(-\lambda)+\lambda+1)\cos(x_n)-(\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l-1}$$
$$-(1-\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^l-(-\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l+1},n=0,\cdots,M,$$

the third order of accuracy difference scheme for the approximate solution of the boundary value problem (3.3).

By using (2.5) and second order of accuracy in x approximation of A, we get the following values of p in grid points

$$p_n=-\frac{\left(\left(\varphi_{n+1}-v_{n+1}^0\right)-2\left(\varphi_n-v_n^0\right)+\left(\varphi_{n-1}-v_{n-1}^0\right)\right)}{h^2} \tag{3.7}$$

$+\left(\varphi_n - v_n^0\right), n = 1, \cdots, M-1.$

We can rewrite difference scheme (3.5) in the matrix form

$$AV_{n+2} + BV_{n+1} + CV_n + DV_{n-1} + EV_{n-2} = I\theta_n,\ n = 2, \cdots, M-2, \tag{3.8}$$

$$-3V_0 + 4V_1 - V_2 = -3V_M + 4V_{M-1} - V_{M-2} = \overrightarrow{0},$$

$$10V_0 - 15V_1 + 6V_2 - V_3 = \overrightarrow{0}, 10V_M - 15V_{M-1} + 6V_{M-2} - V_{M-3} = \overrightarrow{0}$$

Here, I is the $(N+1)\times(N+1)$ identity matrix, A, B, C, D, E are $(N+1)\times(N+1)$ square matrices, θ_n is $(N+1)\times 1$ a column matrix which are defined by the following formulas

$$A = E = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.9}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & y & z & q & 0 & \cdots & 0 & 0 & 0 & 0 \\ r & c & r & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & r & c & r & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r & c & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & c & r & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & r & c & r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & r & c & r \\ 0 & 0 & 0 & 0 & \cdots & 0 & y & z & q & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = D = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.10}$$

$$a = \frac{\tau^2}{12h^4},\ b = -\frac{1}{h^2} - \frac{\tau^2}{3h^4} - \frac{\tau^2}{6h^2}, c = 1 + \frac{2}{\tau^2} + \frac{2}{h^2} - \frac{\tau^2}{12}\left(\frac{6}{h^4} + \frac{4}{h^2} + 1\right), r = -\frac{1}{\tau^2},$$

$$y = -(\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2), z = -(1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2), q = -(-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2),$$

$$\theta_n = \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix}, \theta_n^k = (\exp(-t_k) + 2t_k)\cos(x_n) + \frac{\tau^2}{12}(\exp(-t_k) + 4t_k)\cos(x_n)),$$

$$k = 1, \cdots, N-1,\ n = 1, \cdots, M-1, \theta_n^0 = (1 - \exp(-\lambda) - \lambda)\cos(x_n),$$

$$\theta_n^N = (\exp(-t_N) - \exp(-\lambda) + t_N - \lambda)\cos(x_n), n = 1, \cdots, M-1,$$

and

$$V_s = \begin{bmatrix} v_s^0 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1}, s = n-1, n, n+1.$$

For solving (3.8) we use modified Gauss elimination method ([30]). We seek solution of (3.8) by the formula

$$V_n = \alpha_n V_{n+1} + \beta_n V_{n+2} + \gamma_n, n = M-2, \cdots, 0,$$

where $\alpha_n, \beta_n (n = 0, \cdots, M-2)$ are $(N+1)\times(N+1)$ square matrices and γ_n $(n = 0, \cdots, M-2)$ are $(N+1)\times 1$ column matrices. For the solution of difference equation (3.8) we need to use the following formulas for $\alpha_n, \beta_n, \gamma_n$

$$F_n = (C_n + D_n\alpha_{n-1} + E_n\beta_{n-2} + E_n\alpha_{n-2}\alpha_{n-1}), n = 2, \cdots, M-4.$$

$$\alpha_n = -F_n^{-1}(B_n + D_n\beta_{n-1} + E_n\alpha_{n-2}\beta_{n-1}), \beta_n = -F_n^{-1}A_n,$$

$$\gamma_n = -F_n^{-1}(R\varphi_n - D_n\gamma_{n-1} - E_n\alpha_{n-2}\gamma_{n-1} - E_n\gamma_{n-2}).$$

where

$$\alpha_0 = \frac{4}{3}I, \beta_0 = -\frac{1}{3}I, \alpha_1 = \frac{8}{5}I, \beta_1 = -\frac{3}{5}I, \alpha_{M-2} = 4I, \beta_{M-2} = -3I,$$

$$\alpha_{M-3} = \frac{8}{3}I, \beta_{M-3} = -\frac{5}{3}I,$$

and $\gamma_0, \gamma_1, \gamma_{M-2}, \gamma_{M-3}$ are the $(N+1)\times 1$ zero column vector. For calculation of V_M and V_{M-1} we can get formula

$$V_M = (S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1}(Q_1 - S_{12}S_{22}^{-1}Q_2), V_{M-1} = S_{22}^{-1}(G_2 - S_{21}V_M),$$

where

$$S_{11} = -3A_{M-2} - 8B_{M-2} - 8C_{M-2}\alpha_{M-3} - 3C_{M-2}\beta_{M-3},$$

$$S_{12} = 4A_{M-2} + 9B_{M-2} + 9C_{M-2}\alpha_{M-3} + 4C_{M-2}\beta_{M-3},$$

$$S_{21} = -3B_{M-1} - 8C_{M-1}, S_{22} = A_{M-1} + 4B_{M-1} + 9C_{M-1},$$

$$Q_1 = I\theta_{M-2} - C_{M-2}\gamma_{M-3}, Q_2 = I\theta_{M-1}.$$

For difference scheme (3.6), we get the following matrix form:

$$AW_{n+2} + BW_{n+1} + CW_n + DW_{n-1} + EW_{n-2} = I\eta_n, \ n = 2, \cdots, M-2, \tag{3.11}$$

$$-3W_0 + 4W_1 - W_2 = -3W_M + 4W_{M-1} - W_{M-2} = \overrightarrow{0},$$

$$10W_0 - 15W_1 + 6W_2 - W_3 = \overrightarrow{0}, 10W_M - 15W_{M-1} + 6W_{M-2} - W_{M-3} = \overrightarrow{0}.$$

Here, A, B, D, E are $(N+1)\times(N+1)$ which are defined by (3.9),(3.10), and C is the following matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ r & c & r & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & r & c & r & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r & c & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & r & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & r & c & r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & r & c & r \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.12}$$

$$\eta_n = \begin{bmatrix} \eta_n^0 \\ \vdots \\ \eta_n^N \end{bmatrix},$$

$$\eta_n^0 = (\exp(-t_l) + t_l + 1)\cos(x_n) - (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)V_n^{l-1} - (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)V_n^l$$

$$(-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)V_n^{l+1}, n = 0, \cdots, M,$$

$$\eta_n^N = (\exp(-t_l) + t_l + 1)\cos(x_n) - (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)V_n^{l-1}$$

$$-(1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)V_n^l - (-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)V_n^{l+1}$$

$$\eta_n^k = p(x_n) + \frac{\tau^2}{6}p(x_n), k = 1, \cdots, N-1, \ n = 1, \cdots, M-1,$$

$$W_s = \begin{bmatrix} w_s^0 \\ \vdots \\ w_s^N \end{bmatrix}_{(N+1)\times 1}, s = n-1, n, n+1.$$

Second, we return again to the inverse problem (3.1). Applying (3.4) and fourth order approximation in t, we get, respectively,

$$-\frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} - \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + v_n^k \tag{3.13}$$

$$-\frac{\tau^2}{12}\left[-\frac{1}{h^2}\left(-\frac{v_{n+2}^k - 2v_{n+1}^k + v_n^k}{h^2} + v_{n+1}^k\right) + \frac{2}{h^2}\left(-\frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + v_n^k\right)\right.$$

$$\left.-\frac{1}{h^2}\left(\frac{v_n^k - 2v_{n-1}^k + v_{n-2}^k}{h^2} + v_{n-1}^k\right) - \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + v_n^k\right]$$

$$= (\exp(-t_k) + 2t_k)\sin(x_n) + \frac{\tau^2}{12}(\exp(-t_k) + 4t_k)\sin(x_n)),$$

$$k = 1, \cdots, N-1, \ n = 2, \cdots, M-2,$$

$$-3v_0^k + 4v_1^k - v_2^k = -3v_M^k + 4v_{M-1}^k - v_{M-2}^k = 0,$$

$$10v_0^k - 15v_1^k + 6v_2^k - v_3^k = 10v_M^k - 15v_{M-1}^k + 6v_{M-2}^k - v_{M-3}^k = 0, k = 0, \cdots, N,$$

$$v_n^0 - (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_n^{l-2} - (-\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 + \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_n^{l-1}$$

$$-(1 - (\frac{\lambda}{\tau} - l)^2)v_n^l - (\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 - \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_n^{l+1}$$

$$-(-\frac{1}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_n^{l+2} = (1 - \exp(-\lambda) - \lambda)\sin(x_n), n = 0, \cdots, M,$$

$$v_n^N - (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_n^{l-2} - (-\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 + \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_n^{l-1}$$

$$-(1 - (\frac{\lambda}{\tau} - l)^2)v_n^l - (\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 - \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_n^{l+1}$$

$$-(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l+2}=(\exp(-t_N)-\exp(-\lambda)+t_N-\lambda)\sin(x_n), n=0,\cdots,M$$

the fourth order of accuracy in t and second order accuracy in x difference scheme for the approximate solution of the nonlocal boundary value problem (3.2), and

$$-\frac{w_n^{k+1}-2w_n^k+v_n^{k-1}}{\tau^2}-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k \tag{3.14}$$

$$-\frac{\tau^2}{12}\left[-\frac{1}{h^2}\left(-\frac{w_{n+2}^k-2w_{n+1}^k+w_n^k}{h^2}+w_{n+1}^k\right)-(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l+2}\right.$$

$$+\frac{2}{h^2}\left(-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k\right)-\frac{1}{h^2}\left(\frac{w_n^k-2w_{n-1}^k+w_{n-2}^k}{h^2}+w_{n-1}^k\right)$$

$$\left.-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k\right]=p(x_n)+\frac{\tau^2}{6}p(x_n),$$

$$-(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l+1}, k=1,\cdots,N-1,\ n=2,\cdots,M-2,$$

$$-3w_0^k+4w_1^k-w_2^k=-3w_M^k+4w_{M-1}^k-w_{M-2}^k=0,$$

$$10w_0^k-15w_1^k+6w_2^k-w_3^k=10w_M^k-15w_{M-1}^k+6w_{M-2}^k-w_{M-3}^k=0, k=0,\cdots,N,t$$

$$w_n^0=(\exp(-\lambda)+\lambda+1)\cos(x_n)-(\frac{1}{12}(\frac{\lambda}{\tau}-l)-\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l-2}-(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2$$

$$+\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l-1}-(1-(\frac{\lambda}{\tau}-l)^2)v_n^l-(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l+1}$$

$$-(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l+2},$$

$$w_n^N=(\exp(-\lambda)+\lambda+1)\cos(x_n)-(\frac{1}{12}(\frac{\lambda}{\tau}-l)-\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l-2}-(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2$$

$$+\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l-1}-(1-(\frac{\lambda}{\tau}-l)^2)v_n^l-(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l+1}$$

$$-(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l+2}.$$

the fourth order of accuracy in t and second order accuracy in x difference scheme for the approximate solution of the boundary value problem (3.3).

For the difference scheme (3.13) we have again matrix form (3.8), where square matrices A, B, D, E are defined by (3.9),(3.10) and C is the following matrix

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & d & e & g & y & z & 0 & \cdots & 0 & 0 & 0 & 0 \\ r & c & r & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & r & c & r & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r & c & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & r & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & r & c & r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & r & c & r \\ 0 & 0 & 0 & 0 & \cdots & 0 & d & e & g & y & z & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Here

$$d=-(\frac{1}{12}(\frac{\lambda}{\tau}-l)-\frac{1}{12}(\frac{\lambda}{\tau}-l)^3, e=-(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2+\frac{1}{6}(\frac{\lambda}{\tau}-l)^3),$$

$$g=-(1-(\frac{\lambda}{\tau}-l)^2), y=-(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3),$$

$$z=-(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3).$$

Difference scheme (3.13) can be rewritten in matrix form (3.11), where A, B, D, E are defined by (3.9),(3.10), C is defined by (3.12) and η_n is defined by formulas

$$\eta_n=\begin{bmatrix}\eta_n^0\\ \vdots \\ \eta_n^N\end{bmatrix}, n=0,\cdots,M,$$

$$\eta_n^0=(\exp(-\lambda)+\lambda+1)\cos(x_n)-(\frac{1}{12}(\frac{\lambda}{\tau}-l)-\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l-2}$$
$$-(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2+\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)V_n^{l-1}-(1-(\frac{\lambda}{\tau}-l)^2)V_n^l$$
$$-(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)V_n^{l+1}-(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)V_n^{l+2},$$
$$\eta_n^N=(\exp(-\lambda)+\lambda+1)\cos(x_n)-(\frac{1}{12}(\frac{\lambda}{\tau}-l)-\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)V_n^{l-2}-(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2,$$
$$\eta_n^k=p(x_n)+\frac{\tau^2}{6}p(x_n),\ k=1,\cdots,N-1,\ n=1,\cdots,M-1,$$

Now we give the results of the numerical analysis using by MATLAB programs. The numerical solutions are recorded for different values of N and M. v_n^k represents the numerical solution of difference scheme for nonlocal boundary value problem (3.2) at (t_k, x_n) and u_n^k represents the numerical solution u of inverse problem (3.1) at the same point and p_n represents the numerical solution p of inverse problem at x_n. For their comparison, the error computed by

$$Ev_M^N=\max_{1\le k\le N-1}(\sum_{n=1}^{M-1}\left|v(t_k,x_n)-v_n^k\right|^2h)^{\frac{1}{2}}$$

$$Eu_M^N=\max_{1\le k\le N-1}(\sum_{n=1}^{M-1}\left|u(t_k,x_n)-u_n^k\right|^2h)^{\frac{1}{2}}, Ep_M=(\sum_{n=1}^{M-1}\left|p(x_n)-p_n\right|^2h)^{\frac{1}{2}}.$$

Tables 1-3 contain the numerical results for $N=6, M=108$; $N=10, M=300$. Hence, third and fourth order of accuracy difference schemes are more accurate comparing with the second order of accuracy difference schemes (ADS). Table1 gives the error between the exact solution and solutions derived by difference schemes for nonlocal problem. Table 2 presents error between the exact p solution and approximate p derived by difference schemes. Table 3 includes the error between the exact u solution and solutions derived by difference schemes.

Table 1. Error Ev_M^N

Difference Schemes for v	N=6,M=108	N=10,M=300
Second order ADS	0.013152	0.0039816
Third order ADS	0.0011632	2.28×10^{-4}
Fourth order ADS	1.45×10^{-4}	1.04×10^{-5}

Table 2. Error Ep_M

Calculation of p	N=6,M=108	N=10,M=300
Second order ADS	0.025951	0.0079236
Third order ADS	0.0022861	4.53×10^{-4}
Fourth order ADS	8.73×10^{-4}	1.20×10^{-4}

Table 3. Error Eu_M^N

Difference Schemes for u	N=6,M=108	N=10,M=300
Second order ADS	0.0025445	7.55×10^{-4}
Third order ADS	1.39×10^{-4}	3.76×10^{-5}
Fourth order ADS	9.95×10^{-5}	9.63×10^{-6}

4. Conclusion

In this paper, inverse problem for the multidimensional elliptic equation with Dirichlet-Neumann conditions is considered. The third and fourth orders of accuracy difference schemes for approximate solutions of this problem are presented. Theorems on the stability, almost coercive stability and coercive stability estimates for the solutions of difference schemes for multidimensional elliptic equation are proved. Numerical results in a two dimensional case are given. As it can be seen from Tables 1-3, the third and fourth orders of accuracy difference schemes are more accurate comparing with the second order of accuracy difference scheme.

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