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Abstract. In this work, we consider approximation of nonlocal reverse parabolic problem with second kind boundary and integral conditions. Stability and coercive stability estimates for solution of difference scheme are obtained. Some illustrations of numerical results in examples of one and two dimensional reverse parabolic equations are presented.

INTRODUCTION

Some nonlocal boundary value problems for reverse parabolic differential equation are studied in [1-5] (see also references therein). Let Ω be unit open cube in R^n with boundary $\partial\Omega$, $\overline{\Omega} = S\Omega \cup \partial\Omega$, and

$$a_i : \Omega \to R, \psi : \overline{\Omega} \to R, \mu : [0,1] \to R, f : (0,1)\times\Omega \to R$$

be given functions, σ is known number $(\sigma > 0)$, $\forall x \in \Omega,\ \forall i = 1,...,n, a_i(x) \ge a_0 > 0$.

In [4], nonlocal boundary value problem for multidimensional reverse parabolic equation with second kind boundary and integral conditions

$$\begin{cases} u_t(x,t) + \sum\limits_{r=1}^{n} \left(a_r(x)u_{x_r}(x,t)\right)_{x_r} - \sigma u(x,t) = f(x,t), x = (x_1,..,x_n) \in \Omega, 0 < t < 1, \\ u(x,1) = \int\limits_0^1 \mu(s)u(x,s)ds + \psi(x),\ x \in \overline{\Omega}, \\ \frac{\partial u}{\partial \vec{n}}(x,t) = 0,\ x \in \partial\Omega, 0 \le t \le 1 \end{cases} \tag{1}$$

was investigated. In the paper [5], approximation for multidimensional reverse parabolic differential equation with integral and Dirichlet boundary conditions was studied. Our purpose in this study is to construct stable difference scheme for nonlocal boundary value problem (1).

Introduce the grid spaces $\widetilde{\Omega}_h$ by

$$\widetilde{\Omega}_h = \{x_m = (h_1m_1, \cdots, h_nm_n), m = (m_1, \cdots, m_n), m_i = 0, \cdots, N_i,\ N_ih_i = 1, i = 1, \cdots, n\}.$$

Let us take $\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap \partial\Omega$. Let L_{2h} and W_{2h}^2 be spaces of the grid functions $v^h(x)$ defined on $\widetilde{\Omega}_h$, equipped with the corresponding norms

$$\left\|v^h\right\|_{L_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} |v^h(x)|^2 h_1 \cdots h_n\right)^{1/2},$$

$$\left\|v^h\right\|_{W_{2h}^2} = \left\|v^h\right\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{i=1}^n \left|(v^h(x))_{x_i\overline{x_i},\, m_i}\right|^2 h_i\right)^{1/2}.$$

It is well known that the following operator

$$A_h^x u^h(x) = -\sum_{i=1}^n \left(a_i(x)u_{\overline{x}_i}^h(x)\right)_{x_i, j_i} + \sigma u^h(x),$$

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acting in the space of grid functions $u^h(x)$ which satisfies the condition $Du^h(x)=0$ on $x\in S_h$ is a self-adjoint positive definite operator in L_{2h} (see [5]).

Applying A_h^x, nonlocal boundary value problem (1) can be replaced by the following difference scheme

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}-A_h^x u_{k-1}^h(x)=f^h(t_k,x), \quad 1\le k\le N,\ x\in\widetilde{\Omega}_h, \\ u_N^h(x)=\sum_{j=0}^{N-1}\mu(t_j)u_j^h(x)\tau+\psi^h(x), x\in\widetilde{\Omega}_h. \end{cases} \tag{2}$$

Let τ and $|h|=\sqrt{\sum_{i=1}^{n}h_i^2}$ be small positive real numbers.

Theorem 1 *Solution of difference scheme (2) obeys stability estimate:*

$$\left\|\left\{u_k^h\right\}_1^N\right\|_{\mathscr{C}_\tau(L_{2h})}\le M(\mu,\delta)\left[\left\|\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{\mathscr{C}_\tau(L_{2h})}\right],$$

where positive constant $M(\mu,\delta)$ does depend on μ,δ, but is independent of $\tau,\psi^h(x),f_1^h(x),\cdots,f_{N-1}^h(x)$.

Theorem 2 *Solution of (2) satisfies the next coercive stability estimate:*

$$\left\|\left\{\tau^{-1}(u_k^h-u_{k-1}^h)\right\}_2^N\right\|_{\mathscr{C}_1^\alpha(L_{2h})}\le M(\mu,\delta)\left[\left\|\psi^h\right\|_{W_{2h}^2}+\left\|\left\{f_k^h\right\}_1^N\right\|_{\mathscr{C}_1^\alpha(L_{2h})}\right],$$

where positive constant $M(\delta,\mu)$ does depend on μ,δ, but is independent of $\tau,\psi^h(x),f_1^h(x),\cdots,f_{N-1}^h(x)$.

The proofs of Theorems 1 and 2 are based on coercive stability estimate for the solution of the elliptic difference problem with second kind boundary condition in L_{2h} [7] and the estimates (13) and (17) of paper [5].

NUMERICAL RESULTS

Now, we present numerical results to boundary value problems for reverse parabolic differential equation with second kind boundary and nonlocal integral conditions in 1D and 2D cases.

First, we consider boundary value problem for 1D reverse parabolic differential equation

$$\begin{cases} u_t(x,t)+(1+2x)^2u_{xx}(x,t)+4(1+2x)\ u_x(x,t)-u(x,t)=f(x,t), \\ f(x,t)=-4e^{-2t}\cos x\left(1+x+x^2\right)-4(1+2x)\ e^{-2t}\sin x\ ,\ 0<x<\pi,0<t<1, \\ u(x,1)=\int_0^1 e^{-2\gamma}u(\gamma,x)\ d\gamma+\psi(x) \quad ,\ \psi(x)=\cos x\left(e^{-2}+\frac{e^{-4}}{4}-\frac{1}{4}\right),0\le x\le\pi, \\ u_x(0,t)=0,u_x(\pi,t)=0,\ \ 0\le t\le 1. \end{cases} \tag{3}$$

The exact solution of problem (3) is $u(x,t)=e^{-2t}\cos x$.

Applying (2), we get the following difference scheme

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau}+(1+2x_n)^2\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+4(1+2x_n)\frac{u_{n+1}^{k-1}-u_{n-1}^{k-1}}{2h}-u_n^{k-1}=f(x_n,t_k), \\ k=1,\cdots,N,\ n=1,\cdots,M-1, \\ u_n^N=\sum_{j=0}^{N-1}e^{-2t_j}u_n^j\tau\ +\cos x_n\left(e^{-2}+\frac{e^{-4}}{4}-\frac{1}{4}\right),\ \ n=0,\cdots,M, \\ u_1^k=u_0^k,\ u_M^k=u_{M-1}^k,\ k=0,\cdots,N \end{cases} \tag{4}$$

for approximate solution of problem (3).
System of equation (4) can be rewritten in the matrix form

$$\begin{aligned} & A_n u_{n+1} + B_n u_n + C_n u_{n-1} = I_n \psi_n,\ n = 1, \cdots, M-1, \\ & u_0 = u_1, u_M = u_{M-1}. \end{aligned} \tag{5}$$

Here, ψ_n is $(N+1)\times 1$ column matrix, $I_n = I$ is identity matrix with $(N+1)$ rows, A_n, B_n, C_n are square matrices with $(N+1)$ rows and columns.

$$A_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_n & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & a_n & 0 \end{bmatrix}, C_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_n & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & c_n & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} -e^{-2t_0}\tau & -e^{-2t_1}\tau & -e^{-2t_2}\tau & \cdots & -e^{-2t_{N-2}}\tau & -e^{-2t_{N-1}}\tau & 1 \\ b_n & s & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_n & s & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & s & 0 & 0 \\ 0 & 0 & 0 & \ddots & b_n & s & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_n & s \end{bmatrix},$$

$$\begin{aligned} & a_n = \tfrac{(1+2x_n)^2}{h^2} + \tfrac{2(1+2x_n)}{h},\ b_n = -\tfrac{1}{\tau} - \tfrac{2(1+2x_n)^2}{h^2} - 1, \\ & c_n = \tfrac{(1+2x_n)^2}{h^2} - \tfrac{2(1+2x_n)}{h}, s = \tfrac{1}{\tau}, \\ & \psi_n = \begin{bmatrix} \psi_n^0 & \ldots & \psi_n^N \end{bmatrix}^t, u_j = \begin{bmatrix} u_j^0 & \ldots & u_j^N \end{bmatrix}^t, j = n-1, n, n+1, \\ & \psi_n^0 = \cos x_n \left(e^{-2} + \tfrac{e^{-4}}{4} - \tfrac{1}{4} \right),\ n = 1, \cdots, M-1, \\ & \psi_n^k = f(x_n, t_k),\ k = 1, \cdots, N,\ n = 1, \cdots, M-1. \end{aligned}$$

One can use modified Gauss elimination method for solving difference equations (4) numerically (see [7]).
Approximate solution of (4) is evaluated by

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, n = M-1, \cdots, 1,$$

where

$$u_M = u_{M-1} = \left(A_{M-1} + B_{M-1} + C_{M-1}\alpha_{M-1}\right)^{-1} \left(I_{M-1}\psi_{M-1} - C_{M-1}\beta_{M-1}\right),$$

$\alpha_1 = I$, β_1 is the column vector with $(N+1)$ zeros, α_n $(n = 2, \cdots, M-1)$ are square matrices with $(N+1)$ rows and columns and β_n $(n = 2, \cdots, M-1)$ are column matrices with $(N+1)$ rows such that

$$\alpha_{n+1} = -\left(B_n + C_n\alpha_n\right)^{-1} A_n, \beta_{n+1} = \left(B_n + C_n\alpha_n\right)^{-1} \left(I_n\psi_n - C_n\beta_n\right).$$

Table 1 shows error computed by

$$Eu_M^N = \max_{1 \le k \le N-1} \left(\sum_{n=1}^{M-1} \left(u(x_n, t_k) - u_n^k\right)^2 h \right)^{\frac{1}{2}}$$

for different values of (N, M).

TABLE 1. Error output for DS (4).

(N,M)	**(10,10)**	**(20,20)**	**(40,40)**	**(80,80)**	**(160,160)**
Error	0.6194	0.3819	0.2264	0.1218	0.0630

Second, we consider the next boundary value problem for 2D reverse parabolic differential equation

$$\begin{cases} u_t(x,y,t)+(1+2x)^2u_{xx}(x,y,t)+4(1+2x)u_x(x,y,t) \\ +u_{yy}(x,y,t)-\ u(x,y,t)=f(x,y,t), \\ f(x,y,t)=-\left(4+(1+2x)^2\right)e^{-2t}\cos x\cos y-4(1+2x)e^{-2t}\sin x\cos y, \\ 0<x,y<\pi,\ 0<t<1, \\ u(1,x,y)=\int\limits_0^1 e^{-2\gamma}u(\gamma,x,y)\,d\gamma+\psi(x,y), \\ \psi(x,y)=\cos x\cos y(e^{-2}+\frac{e^{-4}}{4}-\frac{1}{4}),0\le x,y\le\pi, \\ u_x(t,0,y)=0,u_x(t,\pi,y)=0,\ 0\le y\le\pi,0\le t\le 1, \\ u_y(t,x,0)=0,u_y(t,x,\pi)=0,0\le x\le\pi,0\le t\le 1. \end{cases} \tag{6}$$

The exact solution of problem (6) is $u(t,x,y)=e^{-2t}\cos x\cos y$.

Introduce notations:

$$\psi_{m,n}=\psi(x_m,y_n)\,,\ n,m=\overline{0,M},\ f^k_{m,n}=f(t_k,x_m,y_n),\ k=\overline{0,N},\ n,m=\overline{0,M}.$$

Applying (2), we get difference scheme for approximate solution of nonlocal boundary value problem (6) in the next form

$$\begin{cases} \frac{u^k_{m,n}-u^{k-1}_{m,n}}{\tau}+(1+2x_m)^2\frac{u^{k-1}_{m+1,n}-2u^{k-1}_{m,n}+u^{k-1}_{m-1,n}}{h^2}+2(1+2x_m)\frac{u^{k-1}_{m+1,n}-u^{k-1}_{m-1,n}}{h} \\ +\frac{u^{k-1}_{m,n+1}-2u^{k-1}_{m,n}+u^{k-1}_{m,n-1}}{h^2}-u^{k-1}_{m,n}=f^k_{m,n},1\le k\le N-1,1\le n,m\le M-1, \\ u^k_{0,n}=0,\ u^k_{M,n}=0,\ 1\le k\le N-1, \\ u^k_{m,0}=0,\ u^k_{m,M}=0,1\le k\le N-1,1\le n,m\le M-1, \\ u^N_{m,n}=\sum\limits_{j=0}^{N-1}e^{-2t_j}u^j_{m,n}\tau+\cos x_n\cos y_m(e^{-2}+\frac{e^{-4}}{4}-\frac{1}{4}),1\le n,m\le M-1. \end{cases} \tag{7}$$

Difference problem (7) can be rewritten in the matrix form (5). Here A_n, B_n, C_n ,I are $K\times K$ square matrices, and I is the identity matrix, $f^{(n)},u_{n-1},u_n,u_{n+1}$ are the $K\times 1$ column matrices such that $K=(N+1)(M+1)$,

$$f^{(n)}=\left[\ f^0_{0,n}\ \cdots\ f^N_{0,n}\ \ f^0_{1,n}\ \cdots\ f^N_{1,n}\ \cdots\ f^0_{M,n}\ \cdots\ f^N_{M,n}\ \right]^t,$$
$$u_s=\left[\ u^0_{0,s}\ \cdots\ u^N_{0,s}\ \ u^0_{1,s}\ \cdots\ u^N_{1,s}\ \cdots\ u^0_{M,s}\ \cdots\ u^N_{M,s}\ \right]^t,s=n\pm 1,n.$$

Modified Gauss elimination method are used to evaluate solution of (7) via Matlab program. Error output for difference problem (7) is recorded for different values of $(N,M)=(10,10),(20,20),(40,40)$ which are computed by

$$Eu^N_M=\max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\sum_{m=1}^{M-1}(u(x_n,y_m,t_k)-u^k_{n,m})^2h^2\right)^{\frac{1}{2}}.$$

Calculated errors in Tables 1 and 2 point out sufficiently good agreement with theorems on stability solution of corresponding difference schemes of boundary value problems for reverse parabolic differential equation with Neumann boundary and nonlocal integral conditions.

TABLE 2. Error output for DS (7).

(N,M)	**(10,10)**	**(20,20)**	**(40,40)**
Error	0.6075	0.2957	0.1655

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