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Approximate solution of the two-dimensional singular integral equation

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Abstract. Approximate quadrature formulas for the numerical calculation of the two-dimensional Vekua potential and singular integrals are obtained. The mechanical quadrature method for the two-dimensional quasilinear singular integral equation with Vekua operators is described. The numerical results are compared with the exact solution of the integral equation.

Keywords: Two-dimensional singular integral, Singular integral equation, Mechanical quadrature method
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INTRODUCTION

Singular integral equations are used for the solution of widely range of problems of physics and applied mechanics, particularly in the areas of aerodynamics, fluid mechanics, elasticity (see [1-5] and references therein). On the other hand, approximation methods for corresponding two-dimensional singular integral equations remain little studied [5-13].

Consider Vekua potential and singular operators defined by [2]

$$T\rho = -\frac{1}{\pi}\iint\limits_{G}\frac{\rho(\zeta)}{\zeta - z}d\zeta,\ S\rho = -\frac{1}{\pi}\iint\limits_{G}\frac{\rho(\zeta)}{(\zeta - z)^2}d\zeta, \tag{1}$$

respectively.

Our aim in this paper is the construction of quadrature formulas for the numerical calculation of the two-dimensional Vekua potential and singular integrals and their application for solving singular integral equations.

Some properties of the operators T and S are given in [3]. In [7], approximate formula for calculation of the singular integral S in rectangular domain G was suggested.

Let $K = \{z \in \mathbb{C} : \ |z| \le 1\}$ be the unit disk in the complex plane, and $\Gamma = \{z \in \mathbb{C} : \ |z| = 1\}$ be the boundary of K. Case $G = K$ is important for the application. In [5], some grid sets on K were described, but these grid sets do not permit us to get sufficiently small error in approximation of the singular integral.

In this work, we introduce some grid sets and obtain quadrature formulas for numerical the calculation of the two-dimensional Vekua integral operators. In application, we apply these quadrature formulas to solve the two-dimensional quasilinear singular integral equation with above mentioned operators.

APPROXIMATE FORMULA FOR CALCULATION OF $T(\rho|z)$

Introduce the sets of grid points

$$\begin{aligned}&\{r_k \mid r_k = k\tau,\ 1 \le k \le N,\ N\tau = 1\},\\ &\left\{\theta_{k,m} \mid \theta_{k,m} = -\pi + mh_k,\ 0 \le m \le M_k,\ M_k h_k = 2\pi\right\}\ (M_k = 2k+1).\end{aligned}$$

Define

$$D_{k,m} \quad = \quad \left\{\zeta \mid \zeta = re^{i\theta},\ r_k \le r \le r_{k+1}, 1 \le k \le N-1,\ \theta_{k,m} \le \theta \le \theta_{k,m+1},\ 0 \le m \le M_k - 1\right\}.$$

It is clear that area of $D_{k,m}$ is $\pi\tau^2$. Moreover,

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$$\partial D_{k,m}=\Gamma_{km}=\Gamma^1_{km}\cup\Gamma^2_{km}\cup\Gamma^3_{km}\cup\Gamma^4_{km},$$

where

$$\begin{aligned}
\Gamma^1_{km} &= \left\{\zeta \mid \zeta = re^{i\theta_{k,m+1}},\ r_k\le r\le r_{k+1}\right\},\\
\Gamma^2_{km} &= \left\{\zeta \mid \zeta = r_{k+1}e^{i\theta},\ \theta_{k,m+1}\ge\theta\ge\theta_{k,m}\right\},\\
\Gamma^3_{km} &= \left\{\zeta \mid \zeta = re^{i\theta_{k,m}},\ r_{k+1}\ge r\ge r_k\right\},\\
\Gamma^4_{km} &= \left\{\zeta \mid \zeta = r_ke^{i\theta},\ \theta_{k,m}\le\theta\le\theta_{k,m+1}\right\}.
\end{aligned}$$

Let $K_\tau=\{z\in C:\ \ |z|<\tau\}$ be the disk with radius τ. Note that

$$K=K_\tau\bigcup_{k=1}^{N-1}\bigcup_{m=0}^{M_k-1}D_{k,m}.$$

Introduce the following two sets of grid points

$$\Omega^{(1)} = \Omega^{(1)}_{\tau,h}=\left\{z_{k,m},\ z_{k+1,m},\ z_{k+1,m+1},\ z_{k,m+1}\middle|\ 1\le k\le N-1,\ 0\le m\le M_k-1\right\}\cup\{0\},$$

and

$$\Omega^{(2)} = \Omega^{(2)}_{\tau,h}=\left\{z^*_{k,m},\ z^*_{k+1,m},\ z^*_{k+1,m+1},\ z^*_{k,m+1}\ \middle|\ 1\le k\le N-1,\ 0\le m\le M_k-1\right\}\cup\{0\}.$$

Here,

$$\begin{aligned}
&z_{k,m}=r_ke^{i\theta_{k,m}},\ z_{k+1,m}=r_{k+1}e^{i\theta_{k,m}},\\
&z_{k+1,m+1}=r_{k+1}e^{i\theta_{k,m+1}},\ z_{k,m+1}=r_ke^{i\theta_{k,m+1}},\\
&z^*_{k,m}=\left(r_k+\frac{\tau}{2}\right)e^{i\left(\theta_{k,m}+\frac{h_k}{2}\right)},\ z^*_{k+1,m}=\left(r_{k+1}+\frac{\tau}{2}\right)e^{i\left(\theta_{k,m}+\frac{h_k}{2}\right)},\\
&z^*_{k+1,m+1} = \left(r_{k+1}+\frac{\tau}{2}\right)e^{i\left(\theta_{k,m+1}+\frac{h_k}{2}\right)},\ z^*_{k,m+1} = \left(r_k+\frac{\tau}{2}\right)e^{i\left(\theta_{k,m+1}+\frac{h_k}{2}\right)}.
\end{aligned}$$

To get approximate formula for the numerical calculation value of the integral on $\Omega^{(1)}$ and $\Omega^{(2)}$, we approximate function in grid points $\Omega^{(2)}$ and $\Omega^{(1)}$, respectively.

Applying a step-by-step approximation of the function $\rho(z)$, we can write

$$\widehat{\rho}(z)=\rho_{km},\ z\in D_{k,m},\ \rho_{km}=\rho(r_ke^{i\theta_{k,m}}),\ 1\le k\le N-1,\ 0\le m\le M_k-1. \tag{2}$$

Then, we get an approximate formula for the calculation of $T(\rho|z)$:

$$T(\rho|z)\approx T(\widehat{\rho}\mid z)=\sum_{k=1}^{N-1}\sum_{m=0}^{M_k-1}\rho_{km}T_{k,m}(z), \tag{3}$$

where

$$T_{k,m}(z)=-\frac{1}{\pi}\iint\limits_{D_{k,m}}\frac{d\zeta}{\zeta-z}.$$

Using Pompeu formula [3], we can obtain

$$T_{k,m}(z)=\begin{cases}\bar{\bar{z}}+\widetilde{T}_{k,m}(z),\ z\in D_{k,m},\\ \widetilde{T}_{k,m}(z),\ z\notin D_{k,m},\end{cases}$$

where

$$\widetilde{T}_{k,m}(z) = -\frac{1}{2\pi i}\left\{\frac{\overline{z_{k,m+1}}}{z_{k,m+1}}\left[\left(z_{k+1,m+1}-z_{k,m+1}\right)+z\ln\left(\frac{z_{k+1,m+1}-z}{z_{k,m+1}-z}\right)\right]\right.$$
$$+\frac{z_{k+1,m}\overline{z_{k+1,m}}}{z}\ln\left[\frac{\left(z_{k+1,m}-z\right)z_{k+1,m+1}}{\left(z_{k+1,m+1}-z\right)z_{k+1,m}}\right]$$
$$+\frac{\overline{z_{k,m}}}{z_{k,m}}\left[\left(z_{k,m}-z_{k+1,m}\right)+z\ln\left(\frac{z_{k,m}-z}{z_{k+1,m}-z}\right)\right]$$
$$\left.+\frac{z_{k,m+1}\overline{z_{k,m+1}}}{z}\ln\left[\frac{\left(z_{k,m+1}-z\right)z_{k,m}}{\left(z_{k,m}-z\right)z_{k,m+1}}\right]\right\},\ (z\neq 0),$$

$$\widetilde{T}_{k,m}(0) = -\frac{1}{2\pi i}\left\{\frac{\overline{z_{k,m+1}}}{z_{k,m+1}}\left(z_{k+1,m+1}-z_{k,m+1}\right)+\overline{z_{k+1,m}}\frac{z_{k+1,m}-z_{k+1,m+1}}{z_{k+1,m+1}}\right.$$
$$\left.+\frac{\overline{z_{k,m}}}{z_{k,m}}\left(z_{k,m}-z_{k+1,m}\right)+\overline{z_{k,m+1}}\frac{z_{k,m+1}-z_{k,m}}{z_{k,m}}\right\}.$$

Theorem 1. *Let $\rho\in L_2(\overline{K})$. For approximate formula (3) of the potential integral T, the following estimate is satisfied*

$$\|T(\rho\mid z)-T(\widehat{\rho}\mid z)\|_{L_2(\overline{K})}\leq M\tau.$$

APPROXIMATE FORMULA FOR CALCULATION OF $S(\rho|z)$

Applying a step-by-step approximation (2) of the function $\rho(z)$, we can write an approximate formula for calculation of $S(\rho\mid z)$:

$$S(\rho\mid z)\approx S(\widehat{\rho}\mid z)=\sum_{k=1}^{N-1}\sum_{m=0}^{M_k-1}\rho_{km}S_{k,m}(z), \tag{4}$$

where

$$S_{k,m}(z)=-\frac{1}{\pi}\iint_{D_{k,m}}\frac{d\zeta}{(\zeta-z)^2}.$$

Using Pompeu formula [3], we get

$$S_{k,m}(z) = -\frac{1}{2\pi i}\left\{\frac{\overline{z_{k,m+1}}}{z_{k,m+1}}\left[\ln\frac{z_{k+1,m+1}-z}{z_{k,m+1}-z}+z\frac{\left(z_{k+1,m+1}-z_{k,m+1}\right)}{\left(z_{k+1,m+1}-z\right)\left(z_{k,m+1}-z\right)}\right]\right.$$
$$+\frac{z_{k+1,m}\overline{z_{k+1,m}}}{z}\left[\frac{z_{k+1,m}-z_{k+1,m+1}}{\left(z_{k+1,m+1}-z\right)\left(z_{k+1,m}-z\right)}+\frac{1}{z}\ln\frac{\left(z_{k+1,m+1}-z\right)z_{k+1,m}}{\left(z_{k+1,m}-z\right)z_{k+1,m+1}}\right]$$
$$+\frac{\overline{z_{k+1,m}}}{z_{k+1,m}}\left[\ln\frac{z_{k,m}-z}{z_{k+1,m}-z}+z\frac{\left(z_{k,m}-z_{k+1,m}\right)}{\left(z_{k,m}-z\right)\left(z_{k+1,m}-z\right)}\right]$$
$$\left.+\frac{z_{k,m+1}\overline{z_{k,m+1}}}{z}\left[\frac{z_{k,m+1}-z_{k,m}}{\left(z_{k,m}-z\right)\left(z_{k,m+1}-z\right)}+\frac{1}{z}\ln\frac{\left(z_{k,m}-z\right)z_{k,m+1}}{\left(z_{k,m+1}-z\right)z_{k,m}}\right]\right\},\ (z\neq 0), \tag{5}$$

$$S_{k,m}(0) = -\frac{1}{2\pi i}\left\{\frac{\overline{z_{k,m+1}}}{z_{k,m+1}}\ln\frac{z_{k+1,m+1}}{z_{k,m+1}}+\frac{z_{k+1,m}\overline{z_{k+1,m}}}{2}\left(-\frac{1}{z_{k+1,m}^2}+\frac{1}{z_{k+1,m+1}^2}\right)\right.$$
$$\left.+\frac{\overline{z_{k+1,m}}}{z_{k+1,m}}\ln\frac{z_{k,m}}{z_{k+1,m}}+\frac{z_{k,m+1}\overline{z_{k,m+1}}}{2}\left(-\frac{1}{z_{k,m+1}^2}+\frac{1}{z_{k,m}^2}\right)\right\}. \tag{6}$$

Theorem 2. *Let $\rho \in L_2(\overline{K})$. For approximate formula (4) of the singular integral S, the following estimate is satisfied*

$$\|S(\rho \mid z) - S(\widehat{\rho} \mid z)\|_{L_2(\overline{K})} \leq M\tau.$$

APPROXIMATION OF SINGULAR INTEGRAL EQUATION

In this section, we apply the mechanical quadrature method to obtain numerical solution of the two-dimensional quasilinear singular integral equation. We use quadrature formulas (5) and (6) for calculation of the two-dimensional singular integrals.

Consider the following two-dimensional singular equation

$$f(z) - \mu_1(f|z)S(f|z) - \mu_2(f|z)\overline{S(f|z)} = g(z),\ z \in K, \tag{7}$$

where $\mu_1,\ \mu_2$, and g are given functions, $\mu_K = \sup\limits_K(|\mu_1| + |\mu_2|) < 1$.

Let $l_2(\Omega^{(j)})$ $(j = 1,2)$ be spaces of the grid functions $\rho^{\tau,h}$ defined on $\Omega^{(j)}$, equipped with the norms

$$\left\|\rho^{\tau,h}\right\|_{l_2(\Omega^{(j)})} = \sqrt{\pi}\tau\left[\sum_{k=1}^{N-1}\sum_{m=1}^{M_k-1}|\rho_{k,m}|^2\right]^{1/2}, \tag{8}$$

where $\rho_{k,m} = \rho(r_k e^{i\theta_{k,m}}),\ r_k e^{i\theta_{k,m}} \in \Omega^{(j)}$.

Let $f^{(j)}$ be restriction of the function $f(z)$ on the grid $\Omega^{(j)}$ $(f^{(j)} = f(z),\ z \in \Omega^{(j)})$ and $S^{(j)}f(z) = S(f|z),\ z \in \Omega^{(j)},\ j = 1,2$. Applying approximate formula (4), we have the following difference problem for singular integral equation (7)

$$f^{(2)}(z^*) - \mu_1^{(2)}(f^{(2)}|z^*)S^{(2)}f^{(1)}(z^*) - \mu_2^{(2)}(f^{(2)}|z^*)S^{(2)}f^{(1)}(z^*) = g^{(2)}(z^*),\ z^* \in \Omega^{(2)},$$

$$f^{(1)}(z) - \mu_1^{(1)}(f^{(1)}|z)S^{(1)}f^{(2)}(z) - \mu_2^{(1)}(f^{(1)}|z)S^{(1)}f^{(2)}(z) = g^{(1)}(z),\ z \in \Omega^{(1)}. \tag{9}$$

For solving difference problem (9), we use the following iteration formulas

$$f_{n+1}^{(2)}(z^*) = \mu_1^{(2)}(f_n^{(2)}|z^*)S^{(2)}f_n^{(1)}(z^*) + \mu_2^{(2)}(f_n^{(2)}|z^*)S^{(2)}f_n^{(1)}(z^*) + g^{(2)}(z^*),\ z^* \in \Omega^{(2)},$$

$$f_{n+1}^{(1)}(z) = \mu_1^{(1)}(f_{n+1}^{(1)}|z)S^{(1)}f_{n+1}^{(2)}(z) + \mu_2^{(1)}(f_{n+1}^{(1)}|z)S^{(1)}f_{n+1}^{(2)}(z) + g^{(1)}(z),\ z \in \Omega^{(1)},\ n \geq 0. \tag{10}$$

Theorem 3. *Difference problem (9) has an unique solution $\left(f^{(1)}, f^{(2)}\right)$ on the space of grid points $\left(\Omega^{(1)}, \Omega^{(2)}\right)$. Moreover, sequence $\left\{\left(f_n^{(1)}, f_n^{(2)}\right)\right\}_{n=0}^{\infty}$ defined by (9) has a limit.*

Example 1. *Consider the two-dimensional singular integral equation*

$$f(z) - \frac{|f|}{3(|f|+1)}S(f|z) - \frac{|f|}{3(|f|+1)}\overline{S(f|z)} = z\bar{z} - \frac{|z|^2\left(z^2+\bar{z}^2\right)}{3\left(|z|^2+1\right)},\ z \in K. \tag{11}$$

It is easy to see that $f(z) = z\bar{z}$ is the exact solution of singular integral equation (11) and the condition $\mu_K < 1$ is fulfilled.
Matlab software is used for calculation approximately solution of problem (11). We take $f_0^{(1)}(z) = 1,\ z \in K$ for the initial data.

Table 1 displays error $\left\|f_n^{(1)} - f_{n-1}^{(1)}\right\|_{l_2(\Omega^{(1)})}$ between approximate solution in current and previous iteration in the norm defined by (8). Table 2 shows the between the exact solution and the approximate solution in various iterations using norm (8).

TABLE 1. Error between approximate solutions in consecutive iterations

Iteration (n)	**N=10**	**N=20**	**N=40**	**N=80**
1	1.0804	1.0485	1.0309	1.0216
2	2.9699×10^{-2}	3.1593×10^{-2}	3.3232×10^{-2}	3.4137×10^{-2}
3	1.586×10^{-3}	1.9595×10^{-3}	2.3324×10^{-3}	2.5514×10^{-3}
4	1.14×10^{-4}	1.952×10^{-4}	2.7894×10^{-4}	3.3058×10^{-4}
5	9.4905×10^{-6}	2.3439×10^{-5}	4.1685×10^{-5}	5.4144×10^{-5}

TABLE 2. Error analysis

	N=10	**N=20**	**N=40**	**N=80**
$\left\| f_5^{(1)} - f_{exact}^{(1)} \right\|_{l_2(\Omega^{(1)})}$	7.8358×10^{-3}	4.0659×10^{-3}	2.0808×10^{-3}	1.0545×10^{-3}

CONCLUSION

In the present paper, approximate quadrature formulas for the numerical calculation of the two-dimensional Vekua potential and singular integrals are obtained. Moreover, applying the result of the monograph [5] the high order of accuracy quadrature formulas can be presented. The mechanical quadrature method for two-dimensional quasilinear singular integral equation with Vekua operators is described. The numerical results are compared with the exact solution of the integral equation.

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