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Approximation of the inverse elliptic problem with mixed boundary value conditions

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Abstract. The inverse problem for the multidimensional elliptic equation with Neumann-Dirichlet conditions are presented. For the approximate solution of this inverse problem the first and second order of accuracy in t and in space variables difference schemes are constructed. The stability, almost coercive stability and coercive stability estimates for the solution of these difference schemes are obtained. The algorithm for approximate solution is tested in a two-dimensional inverse problem.

Keywords: Difference scheme, Inverse elliptic problem, Well-posedness, Stability, Almost coercive stability, Coercive stability, Overdetermination
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INTRODUCTION

Theory and methods of solutions of the inverse problems for partial differential equations have been extensively investigated by many researchers (see [1–24] and the bibliography therein).

In the present paper, we consider the inverse problem of finding functions $u(t,x)$ and $p(x)$ for multidimensional elliptic equation with following Dirichlet-Neumann boundary conditions

$$\begin{cases} -u_{tt}(t,x) - \sum\limits_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \sigma u(t,x) = f(t,x) + p(x),\ x = (x_1, \cdots, x_n) \in \Omega,\ 0 < t < T, \\ u(0,x) = \varphi(x),\ u(T,x) = \psi(x),\ u(\lambda,x) = \xi(x),\ x \in \overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial \vec{n}} = 0,\ \ x \in S^1,\ u(t,x) = 0,\ x \in S^2, 0 \le t \le T. \end{cases} \tag{1}$$

Here, $0 < \lambda < T$ and $\sigma > 0$ are given numbers, $a_r(x)$, $(x \in \Omega)$, $\varphi(x), \psi(x), \xi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,T),\ x \in \Omega)$ are given smooth functions and $a_r(x) \ge a > 0$ $(x \in \Omega)$, and $\Omega = (0,\ell) \times \cdots \times (0,\ell)$ is the open cube in the n-dimensional Euclidean space with boundary $S = S^1 \cup S^2$, $\overline{\Omega} = \Omega \cup S$.

The well-posedness and approximation of the inverse problem for the multidimensional elliptic equation with Dirichlet conditions were investigated in [13]. Approximation of the inverse problem for multidimensional elliptic equation with Neumann conditions and the well-posedness of difference problems were investigated in [14]. A third and a fourth order of accuracy difference schemes for these problems were constructed in [15, 16].

In the present study, we construct a first and a second order of accuracy in t and in space variables difference schemes for the approximate solution of inverse problem (1) with Neumann-Dirichlet boundary conditions. We establish the stability, the almost coercive stability and the coercive stability estimates for the solution of these difference schemes. The test example for the two-dimensional inverse problem is given.

DIFFERENCE SCHEMES

The discretization of problem (1) is carried out in two steps. Define the sets

$$\begin{cases} \widetilde{\Omega}_h = \{x_m = (h_1 m_1, \cdots, h_n m_n);\ m = (m_1, \cdots, m_n),\ m_q = 0,\ \cdots,\ M_q,\ h_q M_q = \ell,\ q = 1,\ \cdots,\ n\}, \\ \Omega_h = \widetilde{\Omega}_h \cap \Omega,\ S_h^1 = \widetilde{\Omega}_h \cap S^1,\ S_h^2 = \widetilde{\Omega}_h \cap S^2. \end{cases}$$

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Introduce the Hilbert spaces $L_{2h}=L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2=W_2^2(\widetilde{\Omega}_h)$ of grid functions $\rho^h(x)=\{\rho(h_1m_1,\cdots,h_nm_n)\}$ defined on $\widetilde{\Omega}_h$ equipped with the norms

$$\left\|\rho^h\right\|_{L_{2h}}=\left(\sum_{x\in\widetilde{\Omega}_h}|\rho^h|^2h_1\cdots h_n\right)^{1/2},$$

$$\left\|\rho^h\right\|_{W_{2h}^2}=\left\|\rho^h\right\|_{L_{2h}}+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{q=1}^n\left|(\rho^h)_{x_q}\right|^2h_1\cdots h_n\right)^{1/2}$$
$$+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{q=1}^n\left|(\rho^h(x))_{x_q\overline{x_q},\,m_q}\right|^2h_1\cdots h_n\right)^{1/2}.$$

To the differential operator A^x, assign the difference operator A_h^x , defined by the formula,

$$A_h^xu^h=-\sum_{q=1}^n\left(a_q(x)u_{\overline{x}_q}^h\right)_{x_q,m_q}+\sigma u_{x_q}^h \tag{2}$$

acting in the space of grid functions $u^h(x)$ satisfying the conditions $D^hu^h(x)=0$ for all $x\in S_h^1$ and $u^h(x)=0$ for all $x\in S_h^2$. Here, $D^hu^h(x)$ is an approximation of $\frac{\partial u}{\partial\vec{n}}$. It is known that [25, 26] A_h^x is a self-adjoint positive define operator in $L_2\left(\widetilde{\Omega}_h\right)$.

In the first step, by using A_h^x, for obtaining $u^h(t,x)$ functions, we arrive at problem

$$\begin{cases}-\frac{d^2u^h(t,x)}{dt^2}+A_h^xu^h(t,x)=f^h(t,x)+p^h(x),\ 0<t<T,\ x\in\Omega_h,\\ u^h(0,x)=\varphi^h(x),\ u^h(\lambda,x)=\xi^h(x),\ u^h(T,x)=\psi^h(x),\ x\in\widetilde{\Omega}_h.\end{cases} \tag{3}$$

In the second step, applying the approximation formula

$$u(\lambda,x)=u\left(\left[\frac{\lambda}{\tau}\right]\tau,x\right)+O(\tau)$$

for $u^h(\lambda,x)=\xi^h(x)$, we get the first order of accuracy difference scheme

$$\begin{cases}-\tau^{-2}(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x))+A_h^xu_k^h(x)=\theta_k^h(x)+p^h(x),\\ \theta_k^h(x)=f^h(t_k,x),\ t_k=k\tau,\ 1\le k\le N-1,\ x\in\Omega_h,\\ u_0^h(x)=\varphi^h(x),\ u_N^h(x)=\psi^h(x),\ u_l^h(x)=\xi^h(x),\ x\in\widetilde{\Omega}_h,\ N\tau=T.\end{cases} \tag{4}$$

Here, $l=\left[\frac{\lambda}{\tau}\right]$, $[\cdot]$ is a notation for greatest integer function.

By using the approximation formula

$$u^h(\lambda,x)=u^h(l\tau,x)+\left(\frac{\lambda}{\tau}-l\right)(u^h(l\tau+\tau,x)-u^h(l\tau,x))+O(\tau^2)$$

for $u^h(\lambda,x)=\xi^h(x)$, we get the second order of accuracy difference scheme

$$\begin{cases} -\tau^{-2}(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x))+A_h^x u_k^h(x)=\theta_k^h(x)+p^h(x), \\ \theta_k^h(x)=f^h(t_k,x),\ t_k=k\tau,\ 1\le k\le N-1,\ x\in\widetilde{\Omega}_h, \\ u_0^h(x)=\varphi^h(x),\ u_N^h(x)=\psi^h(x), \\ u_l^h(x)+\left(\frac{\lambda}{\tau}-l\right)\left(u_{l+1}^h(x)-u_l^h(x)\right)=\xi^h(x),\ x\in\widetilde{\Omega}_h,\ N\tau=T. \end{cases} \tag{5}$$

Let H be the Hilbert space. To formulate our result on well-posedness of difference schemes, we give definition of $C([0,T]_\tau,H)$ and $\mathscr{C}_{0T}^{\alpha,\alpha}([0,T]_\tau,H)$ which are the linear spaces of mesh functions $\theta^\tau=\{\theta_k\}_1^{N-1}$ with values in the Hilbert space H. We denote $C([0,T]_\tau,H)$ normed space with the norm

$$\left\|\{\theta_k\}_1^{N-1}\right\|_{C([0,T]_\tau,H)}=\max_{1\le k\le N-1}\|\theta_k\|_H,$$

and $\mathscr{C}_{0T}^{\alpha,\alpha}([0,T]_\tau,H)$ normed space with the norm

$$\left\|\{\theta_k\}_1^{N-1}\right\|_{\mathscr{C}_{0T}^{\alpha,\alpha}([0,T]_\tau,H)}=\left\|\{\theta_k\}_1^{N-1}\right\|_{C([0,T]_\tau,H)}$$
$$+\sup_{1\le k<k+n\le N-1}\frac{(k\tau+n\tau)^\alpha(T-k\tau)^\alpha\|\theta_{k+n}-\theta_k\|_H}{(n\tau)^\alpha}.$$

Theorem 1. *Let τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ be sufficiently small positive numbers. Then, the solutions $\left(\left\{u_k^h\right\}_1^{N-1},p^h\right)$ of difference schemes (4) and (5) obey the following stability estimates:*

$$\left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C([0,T]_\tau,L_{2h})}\le M(\delta)\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\xi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C([0,T]_\tau,L_{2h})}\right],$$

$$\left\|p^h\right\|_{L_{2h}}\le M(\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|\xi^h\right\|_{W_{2h}^2}+\frac{1}{\alpha(1-\alpha)}\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{\mathscr{C}_{0T}^{\alpha,\alpha}([0,T]_\tau,L_{2h})}\right],$$

where $M(\delta)$ is independent of $\tau,\alpha,h,\varphi^h,\psi^h$, ξ^h, and $\left\{f_k^h\right\}_1^{N-1}$.

Theorem 2. *Let τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ be sufficiently small positive numbers. Then, the solutions of difference schemes (4) and (5) obey the following almost coercive stability estimate:*

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2})\right\}_1^{N-1}\right\|_{C([0,T]_\tau,L_{2h})}+\left\|p^h\right\|_{L_{2h}}$$
$$\le M(\delta)\left(\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|\xi^h\right\|_{W_{2h}^2}+\ln\left(\frac{1}{\tau+h}\right)\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C([0,T]_\tau,L_{2h})}\right),$$

where $M(\delta)$ does not depend on $\tau,\alpha,h,\varphi^h,\psi^h$, ξ^h, and $\left\{f_k^h\right\}_1^{N-1}$.

Theorem 3. *Let τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ be sufficiently small positive numbers. Then, the solutions of difference schemes (4) and (5) obey the following coercive stability estimate:*

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2})\right\}_1^{N-1}\right\|_{\mathscr{C}_{0T}^{\alpha,\alpha}([0,T]_\tau,L_{2h})}+\left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{\mathscr{C}_{0T}^{\alpha,\alpha}([0,T]_\tau,W_{2h}^2)}+\left\|p^h\right\|_{L_{2h}}$$

$$\leq M(\delta)\left[\frac{1}{\alpha(1-\alpha)}\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{\mathscr{C}_{0T}^{\alpha,\alpha}([0,T]_\tau,L_{2h})}+\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|\xi^h\right\|_{W_{2h}^2}\right],$$

where $M(\delta)$ is independent of $\tau,\alpha,h,\varphi^h,\psi^h$, ξ^h, and $\left\{f_k^h\right\}_1^{N-1}$.

The proofs of Theorems 1-3 are based on representation formulas for solutions, the symmetry property of operator A_h^x in L_{2h}, and the following theorem on the coercivity estimate for the solution of the elliptic difference problem in L_{2h}.

Theorem 4. *[27] For the solution of the elliptic difference problem*

$$\begin{cases} A_h^x u^h(x)=\omega^h(x),\ x\in\widetilde{\Omega}_h, \\ D^h u^h(x)=0,\ x\in S_h^1, u^h(x)=0,\ x\in S_h^2, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{q=1}^{n}\left\|(u^h)_{x_q,\bar{x}_q,m_q}\right\|_{L_{2h}}\leq M||\omega^h||_{L_{2h}},$$

where M does not depend on h and ω^h.

NUMERICAL RESULTS

For the numerical result, we consider the inverse problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial}{\partial x}\left((2+\cos x)\frac{\partial u(t,x)}{\partial x}\right)+u(t,x)=(3\cos(x)+\cos(2x)+1)t \\ +(2\cos(x)+\cos(2x))\exp(-t), 0<x<\pi, 0<t<T, \\ u(0,x)=2(\cos(x)+1),\ u(T,x)=(\exp(-T)+T+1)(\cos(x)+1),\ 0\leq x\leq\pi, \\ u(\lambda,x)=(\exp(-\lambda)+\lambda+1)(\cos(x)+1),\ 0\leq x\leq\pi,\ \ \lambda=\frac{4T}{5}, \\ u_x(t,0)=0,\ u(t,\pi)=0,\ 0\leq t\leq T, \end{cases} \tag{6}$$

for the elliptic equation. It is easy to see that $u(t,x)=(\exp(-t)+t+1)(\cos(x)+1)$ and $p(x)=3\cos(x)+\cos(2x)+1$ are the exact solutions of (6).

Now, we give the results of the numerical analysis using by MATLAB programs. The numerical solutions are recorded for different values of N and M. Tables 1-2 give the error analysis between the exact solution and solutions derived by difference schemes. Tables 1-2 are constructed for $N=M=20$, 40, 80 and 160. Hence, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme. Table 1 gives the error analysis between the exact solution p and solutions derived by difference schemes in second stage of algorithm.

TABLE 1. Error analysis for p

	N=M=20	**N=M=40**	**N=M=80**	**N=M=160**
Difference scheme (4)	0.46799	0.34451	0.24769	0.17645
Difference scheme (5)	0.066127	0.012292	0.0023387	4.61×10^{-4}

Table 2 gives the error analysis between the exact solution u and solutions derived by first order and second order accuracy of difference schemes.

TABLE 2. Error analysis for u

	N=M=20	N=M=40	N=M=80	N=M=160
Difference scheme (4)	0.13789	0.062908	0.030101	0.014731
Difference scheme (5)	0.01591	0.0018966	2.35×10^{-4}	3.01×10^{-5}

CONCLUSION

In this paper, the inverse problem for the multidimensional elliptic equation with Dirichlet-Neumann conditions is considered. The first and second order of accuracy difference schemes for approximate solutions of this problem are presented. Theorems on the stability, almost coercive stability and coercive stability estimates for solutions of difference schemes for the multidimensional elliptic equation are proved. Numerical results in a two-dimensional case are given. As it can be seen from Tables 1-2, the second order of accuracy difference scheme is more accurate than the first order of accuracy difference scheme.

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REFERENCES

1. A. I. Prilepko, D. G. Orlovsky, and I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, New York, 2000.
2. A. A. Samarskii, and P. N. Vabishchevich, *Numerical Methods for Solving Inverse Problems of Mathematical Physics,* in: Inverse and Ill-Posed Problems Series, Walter de Gruyter, Berlin, 2007.
3. V. V. Soloviev, *Zh. Vychil. Mat.i Mat. Fiziki* **44**, 862-871 (2004).
4. D. G. Orlovskii, *Differential Equations* **44**, 124-134 (2008).
5. D. Orlovsky, and S. Piskarev, *J. Inverse Ill-Posed Probl.* **17**, 765-782 (2009).
6. A. Ashyralyev, and A. S. Erdogan, *Vestnik of Odessa National University, Mathematics and Mechanics* **15**, 129-135 (2010).
7. A. Ashyralyev, and A. S. Erdogan, *Int. J. Math. Comput.* **11**, 73-81 (2011).
8. A. Ashyralyev, A. S. Erdogan, and O. Demirdag, *Appl. Numer. Math.* **62**, 1672-1683 (2012).
9. A. S. Erdogan, and H. Uygun, *Abstr. Appl. Anal.* **2012**, 1-26 (2012), Article ID 276080.
10. N. Topsakal, and R. Amirov, *Math. Phys. Anal. Geom.* **13**, 29-46 (2008)
11. A. Ashyralyev, and A. S. Erdogan, *Appl. Math. Comput.* **226**, 212-228 (2014).
12. C. Ashyralyyev, A. Dural, and Y. Sözen, *AIP Conference Proceedings* **1470**, 102-105 (2012).
13. C. Ashyralyyev, and M. Dedeturk, *Contemporary Analysis and Applied Mathematics* **1**, 132-155 (2013).
14. C. Ashyralyyev, and M. Dedeturk, *Abstr. Appl. Anal.* **2013**, 1-11 (2013), Article ID 548017.
15. C. Ashyralyyev, *Bound. Value Probl.* **2014**, 1-23 (2014), doi: 10.1186/1687-2770-2014-5.
16. C. Ashyralyyev, High order approximation of the inverse elliptic problem with Dirichlet-Neumann conditions, Filomat(in press).
17. A. Ashyralyev, and M. Urun, *Contemporary Analysis and Applied Mathematics* **1**, 156-166 (2013).
18. A. Ashyralyev, and C. Ashyralyyev, *Nonlinear Anal. Model. Control* **19**, 350-366 (2014).
19. D. Orlovsky, and S. Piskarev, *Contemporary Analysis and Applied Mathematics* **1**, 118-131 (2013).
20. A. Ashyralyev, *Ukrainian Math. J.* **62**, 1397-1408 (2011).
21. A. Ashyralyev, and Y. A. Sharifov, *Electron. J. Differential Equations* **80** (2013).
22. Y. S. Eidelman, *Math. Notes* **49**, 535-540 (1991).
23. V.G. Romanov, *Dokl. Math.* **84**, 833-836 (2011).
24. K. Sakamoto, and M. Yamamoto, *Appl. Anal.* **88**, 735-748 (2009).
25. S. G. Krein, *Linear Differential Equations in Banach Space*, Nauka, Moscow, 1966.
26. A. Ashyralyev, and P. E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, Birkhäuser Verlag, Basel, Boston, Berlin, 2004.
27. P. E. Sobolevskii, *Difference Methods for the Approximate Solution of Differential Equations*, Voronezh State University Press, Voronezh, 1975.