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Inverse Neumann problem for an equation of elliptic type

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Abstract. Inverse problem for an elliptic differential equation with Neumann conditions is considered. Stability and coercive stability estimates for the solution of inverse problem with the overdetermination are obtained. The first and second order of accuracy difference schemes are presented. Stability and coercive stability inequalities for these difference schemes are given. In application, inverse problem for the multidimensional elliptic equation is studied. The first and second order of accuracy difference schemes for the multidimensional inverse problem are presented. Well-posedness of both difference problems are established. The results are supported by a numerical example for the two-dimensional elliptic equation.

Keywords: Difference scheme, Inverse elliptic problem, Stability, Coercive stability, Overdetermination
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INTRODUCTION

It is known that many problems in various branches of science lead to inverse problems for partial differential equations (see,[1, 2]). Theory and methods of solutions of inverse problems for partial differential equations have been extensively studied by several researchers (see [1–33] and the literature cited therein).

Consider the inverse problem of finding a function u and an element p for an elliptic equation

$$\begin{cases} -u_{tt}(t)+Au(t)=f(t)+pt, 0<t<T, \\ \\ u_t(0)=\varphi,\ u_t(T)=\psi, u_t(\lambda)=\xi, 0<\lambda<T \end{cases} \tag{1}$$

in an arbitrary Hilbert space H with the self-adjoint positive definite operator A. Here, $f(t)$ is a given smooth function, φ, ψ, and ξ are given elements of H, λ is a known number.

In the present work, we obtain stability and coercive stability estimates for the solution of inverse problem (1) and present the first and second order of accuracy difference schemes for its solution. Stability and coercive stability inequalities for difference problems are established. In application, we consider the following inverse problem of finding functions $u(t,x)$ and $p(x)$ for the multidimensional elliptic equation in $[0,T]\times\Omega$,

$$\begin{cases} -u_{tt}(t,x)-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u(t,x)=f(t,x)+p(x)t, \\ x=(x_1,\cdots,x_n)\in\Omega,\ 0<t<T, \\ \\ u_t(0,x)=\varphi(x), u_t(T,x)=\psi(x), u_t(\lambda,x)=\xi(x), x\in\overline{\Omega}, \\ \\ u(t,x)=0,\ x\in S, 0\le t\le T. \end{cases} \tag{2}$$

Here, $\Omega=(0,\ell)\times\cdots\times(0,\ell)$ is the open cube in the n-dimensional Euclidean space with boundary S, $\overline{\Omega}=\Omega\cup S$, $a_r(x)$ $(x\in\Omega)$, $\varphi(x),\psi(x),\xi(x)$ $(x\in\overline{\Omega})$, and $f(t,x)(t\in(0,T),x\in\Omega)$ are given smooth functions, $0<\lambda<T$ and $\delta>0$ are known numbers, $a_r(x)\ge a>0$ $(x\in\Omega)$.
The stability and coercive stability estimates for the solution of inverse problem (2) are obtained. The first and second order of accuracy difference schemes for the approximate solution of problem (2) are presented. Well-posedness of both difference problems are established. The results are supported by numerical example for the two-dimensional elliptic equation.

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MAIN RESULTS

Let $C_{0T}^{\alpha,\alpha}(H)$ be space obtained by completion of the space of all smooth H-valued functions ρ on $[0,T]$ with the norm

$$\|\rho\|_{C_{0T}^{\alpha,\alpha}(H)}=\|\rho\|_{C(H)}+\sup_{0\le t<t+\tau\le T}\frac{(t+\tau)^{\alpha}(T-t)^{\alpha}\|\rho(t+\tau)-\rho(t)\|_H}{\tau^{\alpha}}.$$

Assume that A is a self-adjoint positive definite operator.

Theorem 1. *Suppose that $\varphi,\psi,\xi\in D(A^{-\frac{1}{2}})$, and $f(t)\in C_{0T}^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $(u(t),p)$ of problem (1) the following stability estimates hold:*

$$\|u\|_{C(H)}\le M\left[\left\|A^{-\frac{1}{2}}\varphi\right\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\left\|A^{-\frac{1}{2}}\xi\right\|_H+\|f\|_{C(H)}\right], \tag{3}$$

$$\left\|A^{-1}p\right\|_H\le M\left[\left\|A^{-\frac{1}{2}}\varphi\right\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\left\|A^{-\frac{1}{2}}\xi\right\|_H+\|f\|_{C(H)}\right], \tag{4}$$

where M does not depend on α, φ,ψ,ξ, and $f(t)$.

Theorem 2. *Assume that $\varphi,\psi,\xi\in D(A^{\frac{1}{2}})$, and $f(t)\in C_{0T}^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $(u(t),p)$ of problem (1) the following coercive inequality holds:*

$$\left\|u''\right\|_{C_{0T}^{\alpha,\alpha}(H)}+\|Au\|_{C_{0T}^{\alpha,\alpha}(H)}+\|p\|_H\le M\left[\frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha,\alpha}(H)}+\left\|A^{\frac{1}{2}}\varphi\right\|_H+\left\|A^{\frac{1}{2}}\psi\right\|_H+\left\|A^{\frac{1}{2}}\xi\right\|_H\right], \tag{5}$$

where M is independent of α, φ,ψ,ξ, and $f(t)$.

Introduce the set of grid points $\{t_k=k\tau,\ 1\le k\le N-1,\ N\tau=T\}$. Let $l=\left[\frac{\lambda}{\tau}\right]$, $[\cdot]$ be the greatest integer function. Applying the approximate formulas

$$\begin{aligned}
u_t(\lambda) &= \frac{u_{l+1}-u_l}{\tau}+o(\tau),\\
u_t(\lambda) &= \frac{3u_{l+1}-4u_l+u_{l-1}}{2\tau}+\left(\frac{\lambda}{\tau}-l\right)\left(\frac{3u_{l+2}-4u_{l+1}+u_l}{2\tau}-\frac{3u_{l+1}-4u_l+u_{l-1}}{2\tau}\right)+o(\tau^2)
\end{aligned}$$

for $u_t(\lambda)=\xi$, problem (1) is replaced by first order of accuracy difference scheme

$$\begin{cases}
-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=\theta_k+pt_k,\ \theta_k=f(t_k),\\
t_k=k\tau,\ 1\le k\le N-1,\\
\frac{u_1-u_0}{\tau}=\varphi,\ \frac{u_N-u_{N-1}}{\tau}=\psi,\ \frac{u_{l+1}-u_l}{\tau}=\xi,
\end{cases} \tag{6}$$

and second order of accuracy difference scheme

$$\begin{cases}
-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=\theta_k+pt_k,\ \theta_k=f(t_k),\\
t_k=k\tau,\ 1\le k\le N-1,\\
\frac{-3u_0+4u_1-u_2}{2\tau}=\varphi,\ \frac{3u_N-4u_{N-1}+u_{N-2}}{2\tau}=\psi,\\
\frac{3u_{l+1}-4u_l+u_{l-1}}{2\tau}+\left(\frac{\lambda}{\tau}-l\right)\left(\frac{3u_{l+2}-4u_{l+1}+u_l}{2\tau}-\frac{3u_{l+1}-4u_l+u_{l-1}}{2\tau}\right)=\xi.
\end{cases} \tag{7}$$

For finding a solution $\{u_k\}_{k=1}^{N-1}$ of difference problems (6) and (7), we apply the substitution

$$u_k=v_k+A^{-1}pt_k, \tag{8}$$

where $\{v_k\}_{k=0}^N$ is the solution of auxiliary nonlocal boundary value difference problem. For $\{v_k\}_{k=0}^N$, we get, respectively, the following first and second orders accuracy auxiliary nonlocal boundary value difference problems

$$\begin{cases} -\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Av_k=\theta_k,\ 1\le k\le N-1, \\ \frac{v_1-v_0}{\tau}-\frac{v_{l+1}-v_l}{\tau}=\varphi-\xi,\ \frac{v_N-v_{N-1}}{\tau}-\frac{v_{l+1}-v_l}{\tau}=\psi-\xi, \end{cases} \tag{9}$$

$$\begin{cases} -\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Av_k=\theta_k,\ 1\le k\le N-1, \\ \frac{-3v_0+4v_1-v_2}{2\tau}-\frac{3v_{l+1}-4v_l+v_{l-1}}{2\tau} \\ -(\frac{\lambda}{\tau}-l)\left(\frac{3v_{l+2}-4v_{l+1}+v_l}{2\tau}-\frac{3v_{l+1}-4v_l+v_{l-1}}{2\tau}\right)=\varphi-\xi, \\ \frac{3v_N-4v_{N-1}+v_{N-2}}{2\tau}-\frac{3v_{l+1}-4v_l+v_{l-1}}{2\tau} \\ -\left(\frac{\lambda}{\tau}-l\right)\left(\frac{3v_{l+2}-4v_{l+1}+v_l}{2\tau}-\frac{3v_{l+1}-4v_l+v_{l-1}}{2\tau}\right)=\psi-\xi. \end{cases} \tag{10}$$

For finding unknown element p, we use formula

$$p=A\xi-Av_t(t_l). \tag{11}$$

So, we will consider the following algorithm for solving of problems (6) and (7) which includes three stages. In the first stage, we consider the auxiliary nonlocal boundary value difference problems (9), (10) and obtain $\{v_k\}_{k=0}^N$.

In the second stage, we put $k=l$ and find $v_t(t_l)$. Then, using (11), we obtain p. In the third stage, we apply formula (8) for obtaining the solution $\{u_k\}_{k=1}^{N-1}$ of difference problems (6), (7), respectively.

Note that $C=\frac{1}{2}(\tau A+\sqrt{4A+\tau^2A^2})$ is a self-adjoint positive definite operator and $R=(I+\tau C)^{-1}$ which is defined on the whole space H is a bounded operator (see [35]) . Here, I is the identity operator.

Theorem 3. *Suppose that $\varphi,\psi,\xi\in H$ and $\{\theta_k\}_{k=1}^{N-1}\in C_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for the solutions $\left(\{u_k\}_{k=1}^{N-1},p\right)$ of difference problems (6) and (7) the following stability estimates*

$$\left\|\{u_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\le M\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{12}$$

$$\left\|A^{-1}p\right\|_H\le M\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right] \tag{13}$$

are satisfied, where M does not depend on $\tau,\alpha,\ \varphi,\psi,\xi$, and $\{\theta_k\}_{k=1}^{N-1}$.
Here, $C_\tau(H)$ and $C_\tau^{\alpha,\alpha}(H)$ are the spaces of all H-valued grid functions $\{\theta_k\}_{k=1}^{N-1}$ in the following norms accordingly

$$\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}=\max_{1\le k\le N-1}\|\theta_k\|_H,$$

$$\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}=\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\sup_{1\le k<k+n\le N-1}\frac{(k\tau+n\tau)^\alpha(T-k\tau)^\alpha\|\theta_{k+n}-\theta_k\|_H}{(n\tau)^\alpha}.$$

Theorem 4. *Assume that $\varphi,\psi,\xi\in D(C)$ and $\{\theta_k\}_{k=1}^{N-1}\in C_\tau^{\alpha,\alpha}(H)\,(0<\alpha<1)$. Then, for solutions $\left(\{u_k\}_{k=1}^{N-1},p\right)$ of difference problems (6) and (7) the coercive stability estimate*

$$\begin{aligned} &\left\|\left\{\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}+\left\|\{Au_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}+\|p\|_H \\ \le\ &M\left[\|C\varphi\|_H+\|C\psi\|_H+\|C\xi\|_H+\frac{1}{\alpha(1-\alpha)}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}\right] \end{aligned} \tag{14}$$

is fulfilled, where M does not depend on $\tau,\alpha,\ \varphi,\psi,\xi$, and $\{\theta_k\}_{k=1}^{N-1}$.

APPLICATION

Note that differential expression [34]

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r}(x))_{x_r} + \delta u(x) \tag{15}$$

defines a self-adjoint positive definite operator A^x acting on $L_2(\overline{\Omega})$ with the domain

$$D(A^x) = \left\{ u(x) \in W_2^2(\overline{\Omega}),\ u(x) = 0 \text{ on } S \right\}.$$

Let H be the Hilbert space $L_2(\overline{\Omega})$. By using abstract Theorems 1 and 2, we get the following theorems about well-posedness of problem (2).

Theorem 5. *Assume that A^x is defined by formula (15), $\varphi, \xi, \psi \in D\left((A^x)^{-\frac{1}{2}}\right), f \in C(L_2(\overline{\Omega}))$. Then, for the solution (u, p) of inverse boundary value problem (2), the stability estimates are satisfied:*

$$\begin{aligned} \|u\|_{C(L_2(\overline{\Omega}))} &\leq M\left[\left\|(A^x)^{-\frac{1}{2}}\varphi\right\|_{L_2(\overline{\Omega})} + \left\|(A^x)^{-\frac{1}{2}}\psi\right\|_{L_2(\overline{\Omega})} + \left\|(A^x)^{-\frac{1}{2}}\xi\right\|_{L_2(\overline{\Omega})} + \|f\|_{C(L_2(\overline{\Omega}))}\right], \\ \left\|(A^x)^{-1}p\right\|_{L_2(\overline{\Omega})} &\leq M\left[\left\|(A^x)^{-\frac{1}{2}}\varphi\right\|_{L_2(\overline{\Omega})} + \left\|(A^x)^{-\frac{1}{2}}\psi\right\|_{L_2(\overline{\Omega})} + \left\|(A^x)^{-\frac{1}{2}}\xi\right\|_{L_2(\overline{\Omega})} + \|f\|_{C(L_2(\overline{\Omega}))}\right], \end{aligned}$$

where M is independent of $\alpha, \varphi(x), \xi(x), \psi(x)$, and $f(t,x)$.

Theorem 6. *Suppose that A^x is defined by formula (15), $\varphi, \psi, \xi \in D\left((A^x)^{\frac{1}{2}}\right), f \in C(L_2(\overline{\Omega}))$. Then, for the solution of inverse boundary value problem (2), coercive stability estimate*

$$\left\|u''\right\|_{C_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} + \|u\|_{C_{0T}^{\alpha,\alpha}(W_2^2(\overline{\Omega}))} + \|p\|_{L_2(\overline{\Omega})}$$

$$\leq M\left[\frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} + \|\varphi\|_{W_2^1(\overline{\Omega})} + \|\psi\|_{W_2^1(\overline{\Omega})} + \|\xi\|_{W_2^1(\overline{\Omega})}\right]$$

holds, where M is independent of $\alpha, \varphi(x), \xi(x), \psi(x)$, and $f(t,x)$.

We discretize problem (2) into two steps. In the first step, we define the grid spaces

$$\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \cdots, h_n m_n); m = (m_1, \cdots, m_n), m_r = 0, \cdots, M_r,\ h_r M_r = \ell, r = 1, \cdots, n\}, \Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.$$

Denote difference operator by

$$A_h^x u^h = -\sum_{r=1}^{n}\left(a_r(x) u_{\overline{x}_r}^h\right)_{x_r, j_r} + \delta u^h,$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator.

Let $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ be spaces of the grid functions $\rho^h(x) = \{\rho(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\|\rho\|_{L_{2h}} = \left(\sum_{x \in \widetilde{\Omega}_h} |\rho^h(x)|^2 h_1 \cdots h_n\right)^{1/2},$$

$$\left\|\rho^h\right\|_{W_{2h}^2} = \left\|\rho^h\right\|_{L_{2h}} + \left(\sum_{x \in \widetilde{\Omega}_h}\sum_{r=1}^{n}\left|\rho_{x_r}^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2} + \left(\sum_{x \in \widetilde{\Omega}_h}\sum_{r=1}^{n}\left|(\rho^h(x))_{x_r \overline{x_r},\, n_r}\right|^2 h_1 \cdots h_n\right)^{1/2}.$$

Applying A_h^x to (6) and (7), we arrive for $u^h(t,x)$ functions at auxiliary nonlocal boundary value problem for a system of ordinary differential equations

$$\begin{cases} -\frac{d^2u^h(t,x)}{dt^2}+A_h^x u^h(t,x)=f^h(t,x)+p^h(x)t,\ 0<t<T,\ x\in\widetilde{\Omega}_h, \\ u_t^h(0,x)=\varphi^h(x),\ u_t^h(\lambda,x)=\xi^h(x), u_t^h(T,x)=\psi(x),\ x\in\widetilde{\Omega}_h. \end{cases} \tag{16}$$

In the second step, problem (16) is replaced by first order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h(x)=\theta_k^h(x)+p^h(x)t_k,\ \theta_k^x(x)=f^h(t_k,x), \\ t_k=k\tau,\ 1\le k\le N-1,\ x\in\widetilde{\Omega}_h, \\ \frac{u_1^h(x)-u_0^h(x)}{\tau}=\varphi^h(x),\ \frac{u_N^h(x)-u_{N-1}^h(x)}{\tau}=\psi^h(x),\ \frac{u_{l+1}^h(x)-u_l^h(x)}{\tau}=\xi^h(x),\ x\in\widetilde{\Omega}_h, \end{cases} \tag{17}$$

and second order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h(x)=\theta_k^h(x)+p^h(x)t_k,\ \theta_k^x(x)=f^h(t_k,x), \\ t_k=k\tau,\ 1\le k\le N-1, x\in\widetilde{\Omega}_h, \\ \frac{-3u_0^h(x)+4u_1^h(x)-u_2^h(x)}{2\tau}=\varphi^h(x), \frac{3u_N^h(x)-4u_{N-1}^h(x)+u_{N-2}^h(x)}{2\tau}=\psi^h(x), \\ \frac{3u_{l+1}^h(x)-4u_l^h(x)+u_{l-1}^h(x)}{2\tau}+\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right) \\ \times\left[\frac{3u_{l+2}^h(x)-4u_{l+1}^h(x)+u_l^h(x)}{2\tau}-\frac{3u_{l+1}^h(x)-4u_l^h(x)+u_{l-1}^h(x)}{2\tau}\right]=\xi^h(x), x\in\widetilde{\Omega}_h. \end{cases} \tag{18}$$

Applying the substitution

$$u_k^h(x)=v_k^h(x)+\left(A_h^x\right)^{-1}p^h(x)A_h^x t_k, \tag{19}$$

we reduce difference problems (17) and (18) to the following auxiliary nonlocal boundary value difference problems

$$\begin{cases} -\frac{v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)}{\tau^2}+A_h^x v_k^h(x)=\theta_k^h(x),\ \ \theta_k^x(x)=f^h(t_k,x), \\ t_k=k\tau,\ 1\le k\le N-1, x\in\widetilde{\Omega}_h, \\ \frac{v_1^h(x)-v_0^h(x)}{\tau}-\frac{v_{l+1}^h(x)-v_l^h(x)}{\tau}=\varphi^h(x)-\xi^h(x), \\ \frac{v_N^h(x)-v_{N-1}^h(x)}{\tau}-\frac{v_{l+1}^h(x)-v_l^h(x)}{\tau}=\psi^h(x)-\xi^h(x), \end{cases} \tag{20}$$

and

$$\begin{cases} -\frac{v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)}{\tau^2}+A_h^x v_k^h(x)=\theta_k^h(x),\ \ \theta_k^x(x)=f^h(t_k,x), \\ t_k=k\tau,\ 1\le k\le N-1, x\in\widetilde{\Omega}_h, \\ \frac{-3v_0^h(x)+4v_1^x(x)-v_2^x(x)}{2\tau}-\frac{3v_{l+1}^h(x)-4v_l^h(x)+v_{l-1}^h(x)}{2\tau} \\ -\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)\left(\frac{3v_{l+2}^h(x)-4v_{l+1}^h(x)+v_l^h(x)}{2\tau}-\frac{3v_{l+1}^h(x)-4v_l^h(x)+v_{l-1}^h(x)}{2\tau}\right)=\varphi^h(x)-\xi^h(x), \\ \frac{3v_N^h(x)-4v_{N-1}^h(x)+v_{N-2}^h(x)}{2\tau}-\frac{3v_{l+1}^h(x)-4v_l^h(x)+v_{l-1}^h(x)}{2\tau} \\ -\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)\left(\frac{3v_{l+2}^h(x)-4v_{l+1}^h(x)+v_l^h(x)}{2\tau}-\frac{3v_{l+1}^h(x)-4v_l^h(x)+v_{l-1}^h(x)}{2\tau}\right)=\psi^h(x)-\xi^h(x), \end{cases} \tag{21}$$

respectively. For finding $p^h(x)$, we use formula

$$p^h(x)=A_h^x\xi^h(x)-A_h^x v_t^h(t_l,x), x\in\widetilde{\Omega}_h. \tag{22}$$

Let τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ be sufficiently small positive numbers.

Theorem 7. *Let τ and $|h|$ be sufficiently small positive numbers. Then, for the solutions of difference schemes (17) and (18) the following stability estimates hold:*

$$\left\| \left\{ u_k^h \right\}_1^{N-1} \right\|_{C_\tau(L_{2h})} \le M \left[\left\| \varphi^h \right\|_{L_{2h}} + \left\| \psi^h \right\|_{L_{2h}} + \left\| \xi^h \right\|_{L_{2h}} + \left\| \left\{ f_k^h \right\}_1^{N-1} \right\|_{C_\tau(L_{2h})} \right],$$

where M is independent of $\tau, \alpha, h, \varphi^h(x), \psi^h(x), \xi^h(x)$ and $\left\{ f_k^h(x) \right\}_1^{N-1}$.

Theorem 8. *Let τ and $|h|$ be sufficiently small positive numbers. Then, for the solutions of difference schemes (17) and (18) the following coercive stability estimate holds:*

$$\begin{aligned} & \left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^k}{\tau^2}) \right\}_1^{N-1} \right\|_{C_\tau^{\alpha,\alpha}(L_{2h})} + \left\| \left\{ u_k^h \right\}_1^{N-1} \right\|_{C_\tau^{\alpha,\alpha}(W_{2h}^2)} + \left\| p^h \right\|_{L_{2h}} \\ \le \quad & M \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^2} + \left\| \xi^h \right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)} \left\| \left\{ f_k^h \right\}_1^{N} \right\|_{C_\tau^{\alpha,\alpha}(L_{2h})} \right], \end{aligned}$$

where M does not depend on $\tau, \alpha, h, \varphi^h(x), \psi^h(x), \xi^h(x)$, and $\left\{ f_k^h(x) \right\}_1^{N-1}$.

NUMERICAL RESULTS

We consider the following inverse problem for an elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = f(t,x) + tp(x), 0 < x < \pi, 0 < t < T, \\ f(t,x) = (\exp(-t) + 2t)\sin(x), \\ u_t(0,x) = 0, 0 \le x \le \pi, \\ u_t(T,x) = (-\exp(-T) + 1)\sin(x), 0 \le x \le \pi, \\ u_t(\lambda,x) = (-\exp(-\lambda) + 1)\sin(x), 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le T, \lambda = \frac{3T}{5}. \end{cases} \tag{23}$$

It is easy to see that $u(t,x) = (\exp(-t) + t + 1)\sin(x)$ and $p(x) = 2\sin(x)$ are the exact solutions of (23).

By using MATLAB software, error for the numerical solutions is recorded for $N = M = 20, 40, 80, 160$.

TABLE 1. Error for p

	N=M=20	**N=M=40**	**N=M=80**	**N=M=160**
Difference scheme (17)	0.078526	0.038285	0.018891	9.3816×10^{-3}
Difference scheme (18)	3.9786×10^{-3}	1.0252×10^{-3}	2.6028×10^{-4}	6.558×10^{-5}

TABLE 2. Error for u

	N=M=20	**N=M=40**	**N=M=80**	**N=M=160**
Difference scheme (17)	0.10724	0.05123	0.025029	0.012369
Difference scheme (18)	7.1349×10^{-3}	1.7948×10^{-3}	4.5038×10^{-4}	1.1282×10^{-4}

CONCLUSION

In this work, we consider the inverse problem for the elliptic differential equation with Neumann conditions. Stability and coercive stability estimates for the solution of the inverse problem with overdetermination are obtained. In application, the inverse problem for the multidimensional elliptic equation is studied. The first and second order difference schemes for the approximate solution of inverse problem is presented. Theorems on the stability and coercive stability estimates for the solutions of difference schemes are obtained. The results are supported by a numerical example for the two-dimensional elliptic equation. As it can be seen from Tables 1-2, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

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