On the problem of determining the parameter of an elliptic equation in a Banach space

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Abstract. The boundary value problem of determining the parameter of an elliptic equation $-u''(t) + Au(t) = f(t) + p \ (0 \leqslant t \leqslant T), \quad u(0) = \varphi, \ u(T) = \psi, \quad u(\lambda) = \xi, \ 0 < \lambda < T$, with a positive operator A in an arbitrary Banach space E is studied. The exact estimates are obtained for the solution of this problem in Hölder norms. Coercive stability estimates for the solution of boundary value problems for multi-dimensional elliptic equations are established.

Keywords: elliptic equations, boundary value problems, positivity, exact estimates, stability.

1 Introduction

Theory and methods of solutions of inverse problems of determining the parameter of a partial differential equations have been extensively studied by several researchers (see [1–18] and references therein). It is important that several inverse problems of determining the parameter of partial differential equations can be reduced to nonlocal boundary value problems for partial differential equations (see [1–11] and the literature cited therein). Well-posedness of nonlocal boundary value problems for elliptic type differential and difference equations was studied in a number of papers (see [19–28] and the bibliography therein). In the present study, we consider the following local boundary value problem for an elliptic equation in an arbitrary Banach space E:

$$\begin{aligned} &-u_{tt}(t) + Au(t) = f(t) + p, \quad 0 < t < T, \\ &u(0) = \varphi, \quad u(T) = \psi, \qquad u(\lambda) = \xi, \quad 0 < \lambda < T, \end{aligned} \tag{1}$$

with a positive operator A and the unknown parameter p. Here $\varphi, \psi, \xi \in D(A)$.

Let us $F(E)$ is the some space of smooth E-valued functions on $[0,T]$. We say that $(u(t),p)$ is the solution of problem (1) in $F(E)\times E_1$ if the following conditions are fulfilled:

(i) $u''(t), Au(t)\in F(E), p\in E_1\subset E$;

(ii) $(u(t),p)$ is satisfies the equation and boundary conditions (1).

Problem (1) was considered in [13]. The solvability and uniqueness under some condition for operator A were proved. The well-posedness of this problem in a Hilbert space with the self-adjoint operator A was studied in the paper [3].

By substituting

$$u(t)=v(t)+A^{-1}p \tag{2}$$

into (1), we can reduce inverse problem to solving auxiliary nonlocal boundary value problem for $v(t)$

$$\begin{aligned} &-v_{tt}(t)+Av(t)=f(t), \quad 0<t<T,\\ &v(0)-v(\lambda)=\varphi-\xi, \qquad v(T)-v(0)=\psi-\varphi. \end{aligned} \tag{3}$$

Here p is the unknown element defined by formula

$$p=A\varphi-Av(0). \tag{4}$$

After substituting representation formula (see [4]) for solution of boundary value problem for elliptic equation with Dirichlet condition to boundary conditions of auxiliary nonlocal boundary value problem (3), we can get representation of its solution:

$$\begin{aligned} v(t)={}&\bigl(I-\mathrm{e}^{-2TB}\bigr)^{-1}\biggl\{\bigl[\mathrm{e}^{-tB}-\mathrm{e}^{-(2T-t)B}\bigr]v(0)+\bigl[\mathrm{e}^{-(T-t)B}-\mathrm{e}^{-(T+t)B}\bigr]v(T)\\ &-\bigl[\mathrm{e}^{-(T-t)B}-\mathrm{e}^{-(T+t)B}\bigr](2B)^{-1}\int_0^T\bigl[\mathrm{e}^{-(T-s)B}-\mathrm{e}^{-(T+s)B}\bigr]f(s)\,\mathrm{d}s\biggr\}\\ &+(2B)^{-1}\int_0^T\bigl[\mathrm{e}^{-|t-s|B}-\mathrm{e}^{-(t+s)B}\bigr]f(s)\,\mathrm{d}s, \end{aligned} \tag{5}$$

where $B=A^{1/2}$,

$$v(T)=v(0)+\psi-\varphi, \tag{6}$$

$$\begin{aligned} v(0)={}&-\bigl(I-\mathrm{e}^{-\lambda B}\bigr)^{-1}\bigl(I-\mathrm{e}^{-(T-\lambda)B}\bigr)^{-1}\bigl(I+\mathrm{e}^{-BT}\bigr)\bigl(\mathrm{e}^{-(T-\lambda)B}+\mathrm{e}^{-TB}\bigr)\\ &\times\left[(2B)^{-1}\int_0^T\bigl(\mathrm{e}^{-(T-s)B}-\mathrm{e}^{-(T+s)B}\bigr)f(s)\,\mathrm{d}s+\psi-\varphi\right]\\ &+\bigl(I-\mathrm{e}^{-\lambda B}\bigr)^{-1}\bigl(I-\mathrm{e}^{-(T-\lambda)B}\bigr)^{-1}\bigl(I+e^{-TB}\bigr)\\ &\times\left[\varphi-\xi+(2B)^{-1}\int_0^T\bigl(\mathrm{e}^{-|\lambda-s|B}-\mathrm{e}^{-(\lambda+s)B}\bigr)f(s)\,\mathrm{d}s\right]. \end{aligned} \tag{7}$$

We denote by $C(E)$, the space of continuous E-valued functions $\rho(t)$ defined on the segment $[0,T]$ and equipped with the norm

$$\|\rho\|_{C(E)} = \max_{0\leqslant t\leqslant T} \|\rho(t)\|_E.$$

We introduce notations for the following operators:

$$L_1 = -\left(I - \mathrm{e}^{-\lambda B}\right)^{-1}\left(I - \mathrm{e}^{-(T-\lambda)B}\right)^{-1}\left(I - \mathrm{e}^{-TB}\right)^{-1}\left(\mathrm{e}^{-(T-\lambda)B} + \mathrm{e}^{-TB}\right),$$
$$L_2 = \left(I - \mathrm{e}^{-\lambda B}\right)^{-1}\left(I - \mathrm{e}^{-(T-\lambda)B}\right)^{-1}\left(I + \mathrm{e}^{-TB}\right),$$

$$K_1(t) = -\left(I + \mathrm{e}^{-TB}\right)^{-1}\left(I - \mathrm{e}^{-(T-t)B}\right)\left(I - \mathrm{e}^{-tB}\right),$$
$$K_2(t) = \left(I - \mathrm{e}^{-2TB}\right)^{-1}\left(\mathrm{e}^{-(T-t)B} - \mathrm{e}^{-(T+t)B}\right).$$

Here $0 \leqslant t \leqslant T$, $0 < \lambda < T$.

Applying (2), (4), (5), (6) and (7), we can obtain the following presentation of the solution $(u(t), p)$ of inverse problem (1):

$$\begin{aligned} u(t) = {} & K_1(t)L_1\left[u(T) - u(0) - (2B)^{-1}\int_0^T \left(\mathrm{e}^{-(T-s)B} - \mathrm{e}^{-(T+s)B}\right)f(s)\,\mathrm{d}s\right] \\ & - K_1(t)L_2\left[u(0) - u(\lambda) + (2B)^{-1}\int_0^T \left(\mathrm{e}^{-|\lambda-s|B} - \mathrm{e}^{-(\lambda+s)B}\right)f(s)\,\mathrm{d}s\right] \\ & + K_2(t)u\left[(T) - u(0) - (2B)^{-1}\int_0^T \left(\mathrm{e}^{-(T-s)B} - \mathrm{e}^{-(T+s)B}\right)f(s)\,\mathrm{d}s\right] \\ & + (2B)^{-1}\int_0^T \left(\mathrm{e}^{-|t-s|B} - \mathrm{e}^{-(t+s)B}\right)f(s)\,\mathrm{d}s + u(0), \end{aligned} \tag{8}$$

$$\begin{aligned} p = {} & Au(0) - L_1\left[Au(T) - Au(0) - \frac{B}{2}\int_0^T \left(\mathrm{e}^{-(T-s)B} - \mathrm{e}^{-(T+s)B}\right)f(s)\,\mathrm{d}s\right] \\ & - L_2\left[Au(0) - Au(\lambda) + \frac{B}{2}\int_0^T \left(\mathrm{e}^{-|\lambda-s|B} - \mathrm{e}^{-(\lambda+s)B}\right)f(s)\,\mathrm{d}s\right]. \end{aligned} \tag{9}$$

Let us $C_{0T}^{\alpha,\alpha}(E) = C_{0T}^{\alpha,\alpha}([0,T],E)$ $(0 < \alpha < 1)$ is the Banach space obtained by completion of the set of smooth E-valued functions $\rho(t)$ on $[0,T]$ in the norm

$$\|\rho\|_{C_{0T}^{\alpha,\alpha}(E)} = \|\rho\|_{C(E)} + \sup_{0\leqslant t<t+\tau\leqslant T} \frac{\|\rho(t+\tau) - \rho(t)\|_E (T-t)^\alpha (t+\tau)^\alpha}{\tau^\alpha}.$$

From positivity of operator A in Banach space E it follows that $B = A^{1/2}$ is strongly positive operator in E. Hence, the operator $-B$ is a generator of an analytic semigroup $\exp\{-tB\}$ ($t \geqslant 0$) with exponentially decreasing norm (see [29]) as $t \to \infty$, i.e., for some $M(B) \in [1, +\infty)$, $\alpha(B) \in (0, +\infty)$ and $t > 0$, the following estimates are valid:

$$\bigl\| \exp(-tB)\bigr\|_{E\to E} \leqslant M(B)\exp\bigl(-\alpha(B)t\bigr), \tag{10}$$

$$\bigl\|tB\exp(-tB)\bigr\|_{E\to E} \leqslant M(B)\exp\bigl(-\alpha(B)t\bigr), \tag{11}$$

$$\bigl\|\bigl(I-\exp(-2TB)\bigr)^{-1}\bigr\|_{E\to E} \leqslant M(B)\bigl(1-\exp\bigl(-2T\alpha(B)t\bigr)\bigr)^{-1}. \tag{12}$$

Let us give lemmas which we need in future.

Lemma 1. *For any* $0 < t < t+\tau < T$ *and* $0 \leqslant \alpha \leqslant 1$*, one has inequality* [29]

$$\bigl\| \exp(-tB) - \exp\bigl(-(t+\tau)B\bigr)\bigr\|_{E\to E} \leqslant M\frac{\tau^{\alpha}}{(t+\tau)^{\alpha}}, \tag{13}$$

where M *does not depend on* α*,* t *and* τ*.*

Lemma 2. *The following inequalities hold*:

$$\|L_1\|_{E\to E} \leqslant M, \qquad \|L_2\|_E \leqslant M, \tag{14}$$

$$\|K_1(t)\|_{E\to E} \leqslant M, \quad \|K_2(t)\|_{E\to E} \leqslant M, \quad 0 \leqslant t \leqslant T, \tag{15}$$

$$\|K_1(t)u\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|u\|_E, \quad \|K_2(t)u\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|u\|_E, \quad 0 \leqslant t \leqslant T. \tag{16}$$

Proof. Inequalities (14) and (15) directly follows from (10), (11), (12) and (13). Let us prove (16). For $0 < t < t+\tau$, we have

$$\begin{aligned}
&\bigl(K_1(t+\tau) - K_1(t)\bigr)u \\
&\quad = -\bigl(I+\mathrm{e}^{-TB}\bigr)^{-1}\bigl[-\bigl(\mathrm{e}^{-(T-t-\tau)B} - \mathrm{e}^{-(T-t)B}\bigr) + \bigl(\mathrm{e}^{-(t+\tau)B} - \mathrm{e}^{-tB}\bigr)\bigr]u, \\
&\bigl(K_2(t+\tau) - K_2(t)\bigr)u \\
&\quad = \bigl(I-\mathrm{e}^{-2TB}\bigr)^{-1}\bigl[\bigl(\mathrm{e}^{-(T-t-\tau)B} - \mathrm{e}^{-(T-t)B}\bigr) - \bigl(\mathrm{e}^{-(t+\tau)B} - \mathrm{e}^{-tB}\bigr)\bigr]u.
\end{aligned}$$

Applying (10), (11), (12) and (13), we can get, for $0 < t < t+\tau$,

$$\begin{aligned}
\bigl\|(K_1\bigl(t+\tau) - K_1(t)\bigr)u\bigr\|_E &\leqslant \bigl\|\bigl(I+\mathrm{e}^{-TB}\bigr)^{-1}\bigr\|_{E\to E}\bigl[\bigl\|\mathrm{e}^{-tB} - \mathrm{e}^{-(t+\tau)B}\bigr\|_{E\to E} \\
&\quad + \bigl\|\mathrm{e}^{-(T-t)B} - \mathrm{e}^{-(T-t-\tau\bigr)B}\bigr\|_{E\to E}\bigr]\|u\|_E \\
&\leqslant M\biggl[\frac{\tau^{\alpha}}{(T-t)^{\alpha}} + \frac{\tau^{\alpha}}{(t+\tau)^{\alpha}}\biggr]\|u\|_E,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\bigl\|\bigl(K_2(t+\tau) - K_2(t)\bigr)u\bigr\|_E &\leqslant \bigl\|\bigl(I-\mathrm{e}^{-2TB}\bigr)^{-1}\bigr\|_E\bigl[\bigl\|\mathrm{e}^{-tB} - \mathrm{e}^{-(t+\tau)B}\bigr\|_{E\to E} \\
&\quad + \bigl\|\mathrm{e}^{-(T-t)B} - \mathrm{e}^{-(T-t-\tau)B}\bigr\|_{E\to E}\bigr]\|u\|_E \\
&\leqslant M\biggl[\frac{\tau^{\alpha}}{(T-t)^{\alpha}} + \frac{\tau^{\alpha}}{(t+\tau)^{\alpha}}\biggr]\|u\|_E.
\end{aligned} \tag{18}$$

From last inequalities, (15) and definition of space $C_{0T}^{\alpha,\alpha}(E)$ we obtain (16). $\square$

In the present paper, the exact estimates for the solution of problem (1) in the Hölder norms are obtained. In applications, the exact estimates are established for the solution of the boundary value problems for multi-dimensional elliptic equations with the parameter.

2 $C_{0T}^{\alpha,\alpha}(E)$-estimates for the solution of (1)

Theorem 1. *Assume that A is a positive operator in Banach space E. Let $u(0), u(\lambda)$, $u(T) \in D(A)$ and $f(t) \in C_{0T}^{\alpha;\alpha}(E)$, $0 < \alpha < 1$. Then the following estimates are satisfied for the solution $(u(t), p)$ of inverse problem* (1) *in $C(E) \times E$:*

$$\|u\|_{C(E)} \leqslant M\left[\|u(0)\|_E + \|u(\lambda)\|_E + \|u(T)\|_E + \|f\|_{C(E)}\right], \tag{19}$$

$$\|A^{-1}p\|_E \leqslant M\left[\|u(0)\|_E + \|u(\lambda)\|_E + \|u(T)\|_E + \|f\|_{C(E)}\right], \tag{20}$$

$$\begin{aligned} \|p\|_E \leqslant M\biggl[&\|Au(0)\|_E + \|Au(\lambda)\|_E + \|Au(T)\|_E \\ &+ \frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha,\alpha}(E)}\biggr], \end{aligned} \tag{21}$$

where M is independent on α, $u(0)$, $u(\lambda)$, $u(T)$ and $f(t)$.

Proof. First, we estimate u. We can rewrite (8) in the following form:

$$\begin{aligned} u(t) &= \bigl(-K_1(t)L_1 - K_1(t)L_2 - K_2(t) + I\bigr)u(0) + K_1(t)L_2u(\lambda) \\ &\quad + \bigl(K_1(t)L_1 + K_2(t)\bigr)u(T) + \bigl[\bigl(-K_1(t)L_1 - K_2(t)\bigr)\mathrm{e}^{-(T-t)B} + I\bigr] \\ &\quad \times (2B)^{-1}\int_0^t \mathrm{e}^{-(t-s)B}\bigl(I - \mathrm{e}^{-2sB}\bigr)f(s)\,\mathrm{d}s \\ &\quad + \bigl(-K_1(t)L_1 - K_2(t)\bigr)(2B)^{-1}\int_t^T \bigl(\mathrm{e}^{-(T-s)B} - \mathrm{e}^{-(T+s)B}\bigr)f(s)\,\mathrm{d}s \\ &\quad + (2B)^{-1}\mathrm{e}^{-2tB}\int_t^T \mathrm{e}^{-(s-t)B}\bigl(I - \mathrm{e}^{-2tB}\bigr)f(s)\,\mathrm{d}s \\ &\quad - K_1(t)L_2(2B)^{-1}\int_0^\lambda \bigl(\mathrm{e}^{-(\lambda-s)B} - \mathrm{e}^{-(\lambda+s)B}\bigr)f(s)\,\mathrm{d}s \\ &\quad - K_1(t)L_2(2B)^{-1}\int_\lambda^T \bigl(\mathrm{e}^{-(s-\lambda)B} - \mathrm{e}^{-(\lambda+s)B}\bigr)f(s)\,\mathrm{d}s \\ &= \sum_{k=1}^{8} Q_k(t). \end{aligned} \tag{22}$$

By using Caushy–Schwarz and triangle inequalities, we have

$$\|Q_1(t)\|_E \leqslant \big[\|K_1(t)\|_{E\to E}\big(\|L_1\|_{E\to E} + \|L_2\|_{E\to E}\big) + \|K_2(t)\|_{E\to E} + 1\big]\|u(0)\|_E,$$
$$\|Q_2(t)\|_E \leqslant \|K_1(t)\|_{E\to E}\|L_2\|_{E\to E}\|u(\lambda)\|_E,$$
$$\|Q_3(t)\|_E \leqslant \big[\|K_1(t)\|_{E\to E}\|L_1\|_{E\to E} + \|K_2(t)\|_{E\to E}\big]\|u(T)\|_E.$$

Applying (14), (15), we get

$$\|Q_1(t)\|_E \leqslant M\|u(0)\|_E, \qquad \|Q_2(t)\|_E \leqslant M\|u(\lambda)\|_E, \qquad \|Q_3(t)\|_E \leqslant M\|u(T)\|_E.$$

Now we estimate $Q_4(t)$. From Caushy–Schwarz and triangle inequalities it follows that

$$\begin{aligned}\|Q_4(t)\|_E &\leqslant \big[\|K_1(t)\|_{E\to E}\|L_1\|_{E\to E} + \|K_2(t)\|_{E\to E}\|\mathrm{e}^{-(T-t)B}\|_{E\to E} + 1\big] \\ &\quad\times \|(2B)^{-1}\|_{E\to E}\int_0^t \|\mathrm{e}^{-(t-s)B}\|_{E\to E}\|I-\mathrm{e}^{-2sB}\|_{E\to E}\,\mathrm{d}s\,\|f\|_{C(E)}.\end{aligned}$$

By using (10), (13)–(15), we obtain that

$$\|Q_4(t)\|_E \leqslant \|f\|_{C(E)}.$$

In a exactly similar manner, we can get inequalities for $Q_5(t)$, $Q_6(t)$:

$$\|Q_5(t)\|_E \leqslant \|f\|_{C(E)}, \qquad \|Q_6(t)\|_E \leqslant \|f\|_{C(E)}.$$

Now let us estimate $Q_7(t)$. It is easy to see that

$$\begin{aligned}\|Q_7(t)\|_E &\leqslant \|K_1(t)\|_{E\to E}\|L_2\|_{E\to E}\|(2B)^{-1}\|_{E\to E}\|(2B)^{-1}\|_{E\to E} \\ &\quad\times \int_0^\lambda \big(\|\mathrm{e}^{-(\lambda-s)B}\|_{E\to E} + \|\mathrm{e}^{-(\lambda+s)B}\|_{E\to E}\big)\,\mathrm{d}s\,\|f\|_{C(E)}.\end{aligned}$$

By using (10), (14), (15), we get

$$\|Q_7(t)\|_E \leqslant \|f\|_{C(E)}.$$

In a similar manner, we can obtain estimate for $Q_8(t)$:

$$\|Q_8(t)\|_E \leqslant \|f\|_{C(E)}.$$

Combining estimates for $Q_k(t)$, $k = 1, \dots, 8$, we obtain estimate (19).

Second, we estimate $A^{-1}p$. We can rewrite (9) in the following form:

$$\begin{aligned}p &= (I + L_1 - L_2)Au(0) - L_2Au(\lambda) - L_1Au(T) \\ &\quad + L_1\frac{B}{2}\int_0^T \big(\mathrm{e}^{-(T-s)B} - \mathrm{e}^{-(T+s)B}\big)f(s)\,\mathrm{d}s \\ &\quad - L_2\frac{B}{2}\int_\lambda^T \big(\mathrm{e}^{-(\lambda-s)B} - \mathrm{e}^{-(\lambda+s)B} + \mathrm{e}^{-(s-\lambda)B} - \mathrm{e}^{-(\lambda+s)B}\big)f(s)\,\mathrm{d}s.\end{aligned} \tag{23}$$

$$
\begin{aligned}
A^{-1}p &= (I + L_1 - L_2)u(0) - L_2 u(\lambda) - L_1 u(T) \\
&\quad + L_1 (2B)^{-1} \int_0^T \bigl(\mathrm{e}^{-(T-s)B} - \mathrm{e}^{-(T+s)B}\bigr) f(s)\,\mathrm{d}s \\
&\quad - L_2 (2B)^{-1} \int_\lambda^T \bigl(\mathrm{e}^{-(\lambda-s)B} - \mathrm{e}^{-(\lambda+s)B} + \mathrm{e}^{-(s-\lambda)B} - \mathrm{e}^{-(\lambda+s)B}\bigr) f(s)\,\mathrm{d}s. \qquad (24)
\end{aligned}
$$

From (24), (10), (11), (14), (15), Caushy–Schwarz and triangle inequalities it follows that

$$
\begin{aligned}
\bigl\|A^{-1}p\bigr\|_E &\leqslant \bigl[1 + \|L_1\|_{E\to E} + \|L_2\|_{E\to E}\bigr] \bigl\|u(0)\bigr\|_E + \|L_2\|_{E\to E} \bigl\|u(\lambda)\bigr\|_E \\
&\quad + \|L_1\|_{E\to E} \bigl\|u(T)\bigr\|_E + \|L_1\|_{E\to E} \bigl\|(2B)^{-1}\bigr\|_{E\to E} \\
&\quad \times \int_0^T \bigl(\bigl\|\mathrm{e}^{-(T-s)B}\bigr\|_{E\to E} + \bigl\|\mathrm{e}^{-(T+s)B}\bigr\|_{E\to E}\bigr)\,\mathrm{d}s\, \|f\|_{C(E)} \\
&\quad + \|L_2\|_{E\to E} \bigl\|(2B)^{-1}\bigr\|_{E\to E} \|f\|_{C(E)} \\
&\quad \times \biggl[\int_0^\lambda \bigl(\bigl\|\mathrm{e}^{-(\lambda-s)B}\bigr\|_{E\to E} + \bigl\|\mathrm{e}^{-(\lambda+s)B}\bigr\|_{E\to E}\bigr)\,\mathrm{d}s \\
&\quad + \int_0^\lambda \bigl(\bigl\|\mathrm{e}^{-(s-\lambda)B}\bigr\|_{E\to E} + \bigl\|\mathrm{e}^{-(\lambda+s)B}\bigr\|_{E\to E}\bigr)\,\mathrm{d}s\biggr] \\
&\leqslant M\bigl(\|u(0)\|_E + \bigl\|u(\lambda)\bigr\|_E + \bigl\|u(T)\bigr\|_E + \|f\|_{C(E)}\bigr).
\end{aligned}
$$

Third, we estimate p. Applying (13)–(15), (23), we show estimate (21):

$$
\begin{aligned}
\|p\|_E &\leqslant \bigl(1 + \|L_1\|_{E\to E} + \|L_2\|_{E\to E}\bigr) \bigl\|Au(0)\bigr\|_E \\
&\quad + \|L_2\|_{E\to E} \bigl\|Au(\lambda)\bigr\|_E + \|L_1\|_{E\to E} \bigl\|Au(T)\bigr\|_E + M\|f\|_{C(E)} \\
&\quad + \|L_2\|_{E\to E} \frac{1}{2} \int_0^T \bigl\|B\bigl(\mathrm{e}^{-(T-s)B} - \mathrm{e}^{-(T+s)B}\bigr)\bigr\|_{E\to E} \bigl\|f(s) - f(T)\bigr\|_E\,\mathrm{d}s \\
&\quad + \|L_1\|_{E\to E} \frac{1}{2} \int_0^\lambda \bigl\|B\bigl(\mathrm{e}^{-(\lambda-s)B} - \mathrm{e}^{-(\lambda+s)B}\bigr)\bigr\|_{E\to E} \bigl\|f(s) - f(\lambda)\bigr\|_E\,\mathrm{d}s \\
&\quad + \|L_1\|_{E\to E} \frac{1}{2} \int_\lambda^T \bigl\|B\bigl(\mathrm{e}^{-(s-\lambda)B} - \mathrm{e}^{-(\lambda+s)B}\bigr)\bigr\|_{E\to E} \bigl\|f(s) - f(\lambda)\bigr\|_E\,\mathrm{d}s \\
&\leqslant M_1\biggl(\bigl\|Au(0)\bigr\|_E + \bigl\|Au(\lambda)\bigr\|_E + \bigl\|Au(T)\bigr\|_E + \frac{1}{\alpha(1-\alpha)} \|f\|_{C_{0T}^{\alpha,\alpha}(E)}\biggr).
\end{aligned}
$$

Therefore, Theorem 1 is proved. □

Now we will study the well-posedness of problem (1) in the space $C_{0T}^{\alpha,\alpha}(E)$.

Theorem 2. *Assume that A is a positive operator in Banach space E. Let $u(0), u(\lambda)$, $u(T) \in D(A)$ and $f(t) \in C_{0T}^{\alpha,\alpha}(E)$, $0 < \alpha < 1$. For the solution $(u(t), p)$ of inverse problem* (1) *in $C_{0T}^{\alpha,\alpha}(E) \times E$, the coercive inequality*

$$\begin{aligned} &\left\|u''\right\|_{C_{0T}^{\alpha,\alpha}(E)} + \|Au\|_{C_{0T}^{\alpha,\alpha}(E)} + \|p\|_E \\ &\quad \leqslant M\left[\left\|Au(0)\right\|_E + \left\|Au(\lambda)\right\|_E + \left\|Au(T)\right\|_E + \frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha,\alpha}(E)}\right] \end{aligned} \tag{25}$$

holds, where M is independent of α, $u(0)$, $u(\lambda)$, $u(T)$ and $f(t)$.

Proof. By using (22), we can get

$$\begin{aligned} Au(t) &= \bigl(-K_1(t)L_1 - K_1(t)L_2 - K_2(t) + I\bigr)Au(0) + K_1(t)L_2Au(\lambda) \\ &\quad + \bigl(K_1(t)L_1 + K_2(t)\bigr)Au(T) + \bigl[\bigl(-K_1(t)L_1 - K_2(t)\bigr)\mathrm{e}^{-(T-t)B} + I\bigr] \\ &\quad \times \frac{B}{2}\int_0^t \mathrm{e}^{-(t-s)B}\bigl(I - \mathrm{e}^{-2sB}\bigr)\,\mathrm{d}s\, f(t)\bigl(-K_1(t)L_1 - K_2(t)\bigr) \\ &\quad \times \frac{B}{2}\int_t^T \bigl(\mathrm{e}^{-(T-s)B} - \mathrm{e}^{-(T+s)B}\bigr)\,\mathrm{d}s\, f(t) \\ &\quad + \frac{B}{2}\mathrm{e}^{-2tB}\int_t^T \mathrm{e}^{-(s-t)B}\bigl(I - \mathrm{e}^{-2tB}\bigr)\,\mathrm{d}s\, f(t) \\ &\quad - K_1(t)L_2\frac{B}{2}\int_0^\lambda \bigl(\mathrm{e}^{-(\lambda-s)B} - \mathrm{e}^{-(\lambda+s)B}\bigr)\,\mathrm{d}s f(t) \\ &\quad - K_1(t)L_2\frac{B}{2}\int_\lambda^T \bigl(\mathrm{e}^{-(s-\lambda)B} - \mathrm{e}^{-(\lambda+s)B}\bigr)\,\mathrm{d}s\, f(t) \\ &\quad + \bigl[\bigl(-K_1(t)L_1 - K_2(t)\bigr)\mathrm{e}^{-(T-t)B} + I\bigr] \\ &\quad \times \frac{B}{2}\int_0^t \mathrm{e}^{-(t-s)B}\bigl(I - \mathrm{e}^{-2sB}\bigr)\bigl(f(s) - f(t)\bigr)\,\mathrm{d}s \\ &\quad + \frac{B}{2}\mathrm{e}^{-2tB}\int_t^T \mathrm{e}^{-(s-t)B}\bigl(I - \mathrm{e}^{-2tB}\bigr)\bigl(f(s) - f(t)\bigr)\,\mathrm{d}s \\ &\quad - K_1(t)L_2\frac{B}{2}\int_0^\lambda \bigl(\mathrm{e}^{-(\lambda-s)B} - \mathrm{e}^{-(\lambda+s)B}\bigr)\bigl(f(s) - f(t)\bigr)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&- K_1(t)L_2\frac{B}{2}\int\limits_{\lambda}^{T}\bigl(\mathrm{e}^{-(s-\lambda)B} - \mathrm{e}^{-(\lambda+s)B}\bigr)\bigl(f(s) - f(t)\bigr)\,\mathrm{d}s \\
&= \sum_{i=1}^{13} S_i(t).
\end{aligned} \tag{26}
$$

Let us estimate $S_i(t)$, $i = 1, \dots, 13$, separately. We start with $S_1(t)$. From esimates (14), (15) it follows that, for $0 \leqslant t \leqslant T$,

$$
\begin{aligned}
\bigl\|S_1(t)\bigr\|_E &\leqslant \bigl(\bigl\|K_1(t)\bigr\|_{E\to E}\|L_1\|_{E\to E} + \bigl\|K_1(t)\bigr\|_{E\to E}\|L_2\|_{E\to E} \\
&\quad + \bigl\|K_1(t)\bigr\|_{E\to E} + 1\bigr)\bigl\|Au(0)\bigr\|_E \\
&\leqslant M\bigl\|Au(0)\bigr\|_E.
\end{aligned}
$$

Further, by using (17), (18) and triangle inequality, we show that, for $0 < t < t+\tau < T$,

$$
\begin{aligned}
&\bigl\|S_1(t+\tau) - S_1(t)\bigr\|_E \\
&\quad \leqslant \bigl(\bigl(\bigl\|K_1(t+\tau)\bigr\|_{E\to E} - \bigl\|K_1(t)\bigr\|_{E\to E}\bigr)\bigl(\|L_1\|_{E\to E} + \|L_2\|_{E\to E}\bigr) \\
&\qquad + \bigl(\bigl\|K_2(t+\tau)\bigr\|_{E\to E} - \bigl\|K_2(t)\bigr\|_{E\to E}\bigr)\bigr)\bigl\|Au(0)\bigr\|_E \\
&\quad \leqslant M\left(\frac{\tau^\alpha}{(T-t)^\alpha} + \frac{\tau^\alpha}{(t+\tau)^\alpha}\right)\bigl\|Au(0)\bigr\|_E.
\end{aligned}
$$

So, we have proved

$$
\|S_1\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\bigl\|Au(0)\bigr\|_E.
$$

In a similar manner, we can establish estimates for $S_2(t)$ and $S_3(t)$:

$$
\|S_2\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\bigl\|Au(\lambda)\bigr\|_E, \qquad \|J_3\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\bigl\|Au(T)\bigr\|_E.
$$

Now let us estimate S_4. It easy to show that

$$
S_4(t) = \frac{1}{2}\bigl[\bigl(-K_1(t)L_1 - K_2(t)\bigr)\mathrm{e}^{-(T-t)B} + I\bigr]\bigl(I - \mathrm{e}^{-tB}\bigr)^2 f(t).
$$

From last one, applying (15) and (10), we can get that, for $0 \leqslant t \leqslant T$,

$$
\begin{aligned}
\bigl\|S_4(t)\bigr\|_E &\leqslant \frac{1}{2}\bigl(\bigl\|K_1(t)\bigr\|_{E\to E}\|L_1\|_{E\to E} + \|K_2(t)\|_{E\to E} + 1\bigr) \\
&\quad + \bigl(1 + 2\bigl\|\mathrm{e}^{-tB}\bigr\|_{E\to E} + \bigl\|\mathrm{e}^{-2tB}\bigr\|_{E\to E}\bigr)\|f\|_E \\
&\leqslant M\|f\|_E \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.
\end{aligned}
$$

From this inequality we have

$$
\max_{0\leqslant t\leqslant T}\bigl\|S_4(t)\bigr\|_E \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.
$$

Now we estimate $S_4(t+\tau) - S_4(t)$. Since

$$\begin{aligned}
&S_4(t+\tau) - S_4(t) \\
&\quad = \frac{1}{2}\bigl(-(K_1(t+\tau) - K_1(t))L_1 - \bigl(K_2(t+\tau) - K_2(t)\bigr)\bigr) \\
&\qquad \times \mathrm{e}^{-(T-t-\tau)B}\bigl(I - \mathrm{e}^{-(t+\tau)B}\bigr)^2 f(t+\tau) \\
&\qquad + \frac{1}{2}\bigl(-K_1(t)L_1 - K_2(t)\bigr)\bigl(\mathrm{e}^{-(T-t-\tau)B} + \mathrm{e}^{-(T+t)B}\bigr)\bigl(I - \mathrm{e}^{-\tau B}\bigr) f(t+\tau) \\
&\qquad - \frac{1}{2}\bigl(\mathrm{e}^{-tB} - \mathrm{e}^{-(t+\tau)B}\bigr)\bigl(2I - \mathrm{e}^{-(t+\tau)B} - \mathrm{e}^{-tB}\bigr) f(t+\tau) \\
&\qquad - \frac{1}{2}\bigl[\bigl(-K_1(t)L_1 - K_2(t)\bigr)\mathrm{e}^{-(T-t)B}\bigl(I - \mathrm{e}^{-tB}\bigr)^2 + \bigl(I - \mathrm{e}^{-tB}\bigr)^2\bigr] \\
&\qquad \times \bigl(f(t) - f(t+\tau)\bigr),
\end{aligned}$$

it follows that

$$\begin{aligned}
&\bigl\|S_4(t+\tau) - S_4(t)\bigr\|_E \\
&\quad \leqslant \frac{1}{2}\bigl(\bigl\|K_1(t+\tau) - K_1(t)\bigr\|_{E\to E}\|L_1\|_{E\to E} + \bigl\|K_2(t+\tau) - K_2(t)\bigr\|_{E\to E}\bigr) \\
&\qquad \times \bigl\|\mathrm{e}^{-(T-t-\tau)B}\bigr\|_{E\to E}\bigl\|I - \mathrm{e}^{-(t+\tau)B}\bigr\|_{E\to E}^2\bigl\|f(t+\tau)\bigr\|_E \\
&\qquad + \frac{1}{2}\bigl(\bigl\|K_1(t)\bigr\|_{E\to E}\|L_1\|_{E\to E} + \bigl\|K_2(t)\bigr\|_{E\to E}\bigr) \\
&\qquad \times \bigl(\bigl\|\mathrm{e}^{-(T-t-\tau)B}\bigr\|_{E\to E} + \bigl\|\mathrm{e}^{-(T+t)B}\bigr\|_{E\to E}\bigr)\bigl\|I - \mathrm{e}^{-\tau B}\bigr\|_{E\to E}\bigl\|f(t+\tau)\bigr\|_E \\
&\qquad + \frac{1}{2}\bigl\|\mathrm{e}^{-tB} - \mathrm{e}^{-(t+\tau)B}\bigr\|_{E\to E}\bigl(2 + \bigl\|\mathrm{e}^{-(t+\tau)B}\bigr\|_{E\to E} + \bigl\|\mathrm{e}^{-tB}\bigr\|_{E\to E}\bigr)\bigl\|f(t+\tau)\bigr\|_E \\
&\qquad + \frac{1}{2}\bigl[\bigl(\bigl\|K_1(t)\bigr\|_{E\to E}\|L_1\|_{E\to E} + \bigl\|K_2(t)\bigr\|_{E\to E}\bigr)\bigl\|I - \mathrm{e}^{-tB}\bigr\|_{E\to E}^2 \\
&\qquad + \bigl\|I - \mathrm{e}^{-tB}\bigr\|_{E\to E}^2\bigr]\bigl\|f(t+\tau) - f(t)\bigr\|_E
\end{aligned}$$

for $0 < t < t+\tau < T$. From last estimate and (10), (13)–(15), (17), (18) we obtain that

$$\bigl\|S_4(t+\tau) - S_4(t)\bigr\|_E \leqslant M\biggl(\frac{\tau^\alpha}{(T-t)^\alpha} + \frac{\tau^\alpha}{(t+\tau)^\alpha}\biggr)\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.$$

Therefore, we have proved

$$\|S_4\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.$$

For S_5, S_6, S_7, S_8, we have

$$\begin{aligned}
S_5(t) &= \frac{1}{2}\bigl[\bigl(-K_1(t)L_1 - K_2(t)\bigr)\bigr]\bigl(I - \mathrm{e}^{-TB}\bigr)^2 f(t), \\
S_6(t) &= \frac{1}{2}\bigl(\mathrm{e}^{-(T+t)B} - \mathrm{e}^{-2tB} + \mathrm{e}^{-(T+3t)B} - \mathrm{e}^{-4tB}\bigr) f(t),
\end{aligned}$$

$$S_7(t) = -\frac{1}{2}K_1(t)L_2\bigl(I - \mathrm{e}^{-\lambda B}\bigr)^2 f(t),$$
$$S_8(t) = -\frac{1}{2}K_1(t)L_2\bigl(\mathrm{e}^{-(T-\lambda)B} - I + \mathrm{e}^{-(T+\lambda)B} + \mathrm{e}^{-2\lambda B}\bigr)f(t).$$

In a similar manner we can show estimates for $S_5(t)$, $S_6(t)$, $S_7(t)$, $S_8(t)$:

$$\|S_5\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}, \qquad \|S_6\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)},$$
$$\|S_7\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}, \qquad \|S_8\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.$$

Now we estimate S_9. We can rewrite S_9 in the form

$$\begin{aligned}
S_9(t) &= \bigl[\bigl(-K_1(t)L_1 - K_2(t)\bigr)\mathrm{e}^{-(T-t)B} + I\bigr]\bigl(I - \mathrm{e}^{-(2T-2t)B}\bigr)^{-1}\bigl(I - \mathrm{e}^{-2TB}\bigr)\\
&\quad\times \bigl(I - \mathrm{e}^{-2TB}\bigr)^{-1}\frac{B}{2}\bigl(I - \mathrm{e}^{-(2T-2t)B}\bigr)\\
&\quad\times \int_0^t \mathrm{e}^{-(t-s)B}\bigl(I - \mathrm{e}^{-2sB}\bigr)\bigl(f(s) - f(t)\bigr)\,\mathrm{d}s\\
&= \bigl[\bigl(-K_1(t)L_1 - K_2(t)\bigr)\mathrm{e}^{-(T-t)B} + I\bigr]\bigl(I - \mathrm{e}^{-(2T-2t)B}\bigr)^{-1}\bigl(I - \mathrm{e}^{-2TB}\bigr)W(t),
\end{aligned}$$

where

$$W(t) = \bigl(I - \mathrm{e}^{-2TB}\bigr)^{-1}\frac{B}{2}\bigl(I - \mathrm{e}^{-(2T-2t)B}\bigr)\int_0^t \mathrm{e}^{-(t-s)B}\bigl(I - \mathrm{e}^{-2sB}\bigr)\bigl(f(s) - f(t)\bigr)\,\mathrm{d}s.$$

For operator $W(t)$, the following estimates are valid (see [29, pp. 213–218, (5.17), (5.19)]):

$$\max_{0\leqslant t\leqslant T}\bigl\|W(t)\bigr\|_E \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}, \tag{27}$$
$$\|W\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}. \tag{28}$$

Applying (10), (12), (14), (15), (27), we can get that, for $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\bigl\|S_9(t)\bigr\|_E &\leqslant \bigl(\bigl(\bigl\|K_1(t)\bigr\|_{E\to E}\|L_1\|_{E\to E} + \bigl\|K_2(t)\bigr\|_{E\to E}\bigr)\bigl\|\mathrm{e}^{-(T-t)B}\bigr\|_{E\to E} + 1\bigr)\\
&\quad\times \bigl\|\bigl(I - \mathrm{e}^{-(2T-2t)B}\bigr)^{-1}\bigr\|_{E\to E}\bigl\|\bigl(I - \mathrm{e}^{-2TB}\bigr)\bigr\|_{E\to E}\bigl\|W(t)\bigr\|_{E\to E}\\
&\leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.
\end{aligned}$$

By using (10), (12), (14), (15), (28), we obtain

$$\begin{aligned}
\|S_9\|_{C_{0T}^{\alpha,\alpha}(E)} &\leqslant \bigl(\bigl(\bigl\|K_1(t)\bigr\|_{E\to E}\|L_1\|_{E\to E} + \bigl\|K_2(t)\bigr\|_{E\to E}\bigr)\bigl\|\mathrm{e}^{-(T-t)B}\bigr\|_{E\to E} + 1\bigr)\\
&\quad\times \bigl\|\bigl(I - \mathrm{e}^{-(2T-2t)B}\bigr)^{-1}\bigr\|_{E\to E}\bigl\|\bigl(I - \mathrm{e}^{-2TB}\bigr)\bigr\|_{E\to E}\|W\|_{C_{0T}^{\alpha,\alpha}(E)}\\
&\leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.
\end{aligned}$$

So, we have showed that

$$\|S_9\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.$$

Estimates S_{10}, S_{11}, S_{12} and S_{13} will be established in the same manner. Hence, we have

$$\|S_{11}\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}, \qquad \|S_{11}\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)},$$
$$\|S_{12}\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}, \qquad \|S_{13}\|_{C_{0T}^{\alpha,\alpha}(E)} \leqslant M\|f\|_{C_{0T}^{\alpha,\alpha}(E)}.$$

Finally, combining estimates S_k, $k = 1, \dots, 13$, and inequality (21), we get estimate

$$\begin{aligned}
&\|Au\|_{C_{0T}^{\alpha,\alpha}(E)} \\
&\quad \leqslant M\left[\big\|Au(0)\big\|_E + \big\|Au(\lambda)\big\|_E + \big\|Au(T)\big\|_E + \frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha,\alpha}(E)}\right].
\end{aligned}$$

By the triangle inequality, this last inequality, (21) and differential equation of problem (1) yields

$$\begin{aligned}
&\big\|u''(t)\big\|_{C_{0T}^{\alpha,\alpha}(E)} \\
&\quad \leqslant M\left[\big\|Au(0)\big\|_E + \big\|Au(\lambda)\big\|_E + \big\|Au(T)\big\|_E + \frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha,\alpha}(E)}\right].
\end{aligned}$$

Therefore, Theorem 2 is proved. □

We denote by $\widetilde{C}_{0T}^{\alpha,\alpha}(E) = \widetilde{C}_{0T}^{\alpha,\alpha}([0,T], E)$, $0 < \alpha < 1$, the Banach space obtained by completion of the set of smooth E-valued functions $\rho(t)$ on $[0,T]$ in the norm

$$\|\rho\|_{\widetilde{C}_{0T}^{\alpha,\alpha}} = \|\rho\|_{C(E)} + \sup_{0\leqslant t<t+\tau\leqslant T} \frac{\|\rho(t+\tau)-\rho(t)\|_E}{\tau^\alpha} \min\bigl\{(T-t)^\alpha,\, (t+\tau)^\alpha\bigr\}.$$

In exactly similar manner as Theorem 1, we can establish the following result.

Theorem 3. *Suppose that A is a positive operator in Banach space E; $u(0), u(\lambda), u(T) \in D(A)$ and $f(t) \in \widetilde{C}_{0T}^{\alpha,\alpha}(E)$, $0 < \alpha < 1$. For the solution $(u(t), p)$ of inverse problem* (1) *in $\widetilde{C}_{0T}^{\alpha,\alpha}(E) \times E$, the coercive inequality*

$$\begin{aligned}
&\|u''\|_{\widetilde{C}_{0T}^{\alpha,\alpha}(E)} + \|Au\|_{\widetilde{C}_{0T}^{\alpha,\alpha}(E)} + \|p\|_E \\
&\quad \leqslant M\left[\big\|Au(0)\big\|_E + \big\|Au(\lambda)\big\|_E + \big\|Au(T)\big\|_E + \frac{1}{\alpha(1-\alpha)}\|f\|_{\widetilde{C}_{0T}^{\alpha,\alpha}(E)}\right]
\end{aligned} \tag{29}$$

holds, where M is independent of α, $u(0)$, $u(\lambda)$, $u(T)$ and $f(t)$.

3 Applications

In this section, we consider applications of abstract Theorems 1 and 2. First, we consider the boundary value problem on the range $\{0 \leqslant t \leqslant T,\ x \in R^n\}$ for $2m$-order multidimensional elliptic equation

$$\begin{aligned} &-u_{tt}(t,x) + \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|} u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta u(t,x) = f(t,x) + p(x), \\ &\quad x \in R^n,\ 0 < t < T, \\ &u(0,x) = \varphi(x), \quad u(T,x) = \psi(x), \quad u(\lambda,x) = \xi(x), \quad x \in R^n, \end{aligned} \tag{30}$$

where $a_r(x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$ are given sufficiently smooth functions and $a_r(x) > 0$, $\delta > 0$ is the sufficiently large number.

Suppose that the symbol

$$B^x(\zeta) = \sum_{|r|=2m} a_r(x)(i\zeta_1)^{r_1} \cdots (i\zeta_n)^{r_n}, \quad \zeta = (\zeta_1, \dots, \zeta_n) \in R^n,$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \tag{31}$$

acting on functions defined on the space R^n, satisfies the inequalities

$$0 \leqslant M_1|\zeta|^{2m} \leqslant (-1)^m B^x(\zeta) \leqslant M_2|\zeta|^{2m} < \infty$$

for $\zeta \neq 0$. So, we have boundary value problem in a Banach space $E = C^\mu(R^n)$ of all continuous bounded functions defined on R^n satisfying a Hölder condition with the indicator $\mu \in (0,1)$ and with a strongly positive operator $A^x = B^x + \delta I$ defined by (31).

Theorem 4. *For the solution of boundary value problem* (30)*, the following coercive estimate is valid*:

$$\begin{aligned} &\|u\|_{C_{0T}^{2+\alpha,\alpha}(C^\mu(R^n))} + \sum_{|r|=2m} \left\| a_r(x) \frac{\partial^{|r|} u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C_{0T}^{\alpha,\alpha}(C^\mu(R^n))} + \|p\|_{C^\mu(R^n)} \\ &\quad \leqslant \frac{M(\mu)}{\alpha(1-\alpha)} \|f\|_{C_{0T}^{2+\alpha,\alpha}(C^\mu(R^n))} + M(\mu) \left[\sum_{|r|=2m} \left\| a_r(x) \frac{\partial^{|r|} \varphi}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C^\mu(R^n)} \right. \\ &\qquad \left. + \sum_{|r|=2m} \left\| a_r(x) \frac{\partial^{|r|} \xi}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C^\mu(R^n)} + \sum_{|r|=2m} \left\| a_r(x) \frac{\partial^{|r|} \psi}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C^\mu(R^n)} \right], \end{aligned}$$

where $0 < \alpha < 1$, $0 < \mu < 1$, $M(\mu)$ *is independent of* α, $\varphi(x)$, $\xi(x)$, $\psi(x)$ *and* $f(t,x)$.

The proof of Theorem 4 is based on the abstract Theorems 1, 2 and the positivity of the operator A^x in $C^\mu(R^n)$, the structure of the fractional spaces $E_\alpha((A^x)^{1/2}, C(R^n))$ [29] and the coercivity inequality for an elliptic operator A^x in $C^\mu(R^n)$ [30].

Second, let $\Omega = (0,1) \times \cdots \times (0,1)$ be the open cube in the n-dimensional Euclidean space with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$, we consider the mixed boundary value problem for multidimensional elliptic equation

$$\begin{aligned} &-u_{tt}(t,x) - \sum_{r=1}^{n} a_r(x) \frac{\partial^2 u}{\partial x_r^2} + \delta u(t,x) = f(t,x) + p(x), \\ &\quad x = (x_1, \ldots, x_n) \in \Omega,\ 0 < t < T, \\ &u(0,x) = \varphi(x), \quad u(T,x) = \psi(x), \quad u(\lambda,x) = \xi(x), \quad x \in \overline{\Omega}, \\ &u(t,x) = 0, \quad x \in S, \end{aligned} \tag{32}$$

where $a_r(x)$ ($a_r(x) > 0$, $x \in \Omega$) and $\varphi(x)$, $\psi(x)$, $\xi(x)$ ($x \in \overline{\Omega}$) are sufficiently smooth functions, $\delta > 0$ is the sufficiently large number.

We denote by $C_{01}^{\beta}(\overline{\Omega})$ ($\beta = (\beta_1, \ldots, \beta_n)$), the Banach spaces of all continuous functions satisfying a Hölder condition with the indicator β and with weight $x_i^{\beta_i}(1 - x_i - h_i)^{\beta_i}$, $0 \leqslant x_i < x_i + h_i \leqslant 1$, $1 \leqslant i \leqslant n$, which equipped with the norm

$$\begin{aligned} \|f\|_{C_{01}^{\beta}(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})} + \sup_{\substack{0 \leqslant x_i < x_i + h_i \leqslant 1 \\ 1 \leqslant i \leqslant n}} \bigl| f(x_1, \ldots, x_n) - f(x_1 + h_1, \ldots, x_n + h_n) \bigr| \\ \times \prod_{i=1}^{n} \left(\frac{x_i}{h_i} \right)^{\beta_i} (1 - x_i - h_i)^{\beta_i}. \end{aligned}$$

It is known that the differential expression

$$A^x u = -\sum_{r=1}^{n} a_r(x) \frac{\partial^2 u}{\partial x_r^2} + \delta u(t,x) \tag{33}$$

defines a positive operator A^x acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $u = 0$ on S. Therefore, using results of abstract Theorems 1 and 2, we can obtain the following statement.

Theorem 5. *For the solution of boundary value problem* (32)*, the following coercive estimate is satisfied*:

$$\begin{aligned} &\|u\|_{C_{0T}^{2+\alpha,\alpha}(C_{01}^{\mu}(\overline{\Omega}))} + \sum_{r=1}^{n} \left\| a_r(x) \frac{\partial^2 u}{\partial x_r^2} \right\|_{C_{0T}^{\alpha,\alpha}(C_{01}^{\mu}(\overline{\Omega}))} + \|p\|_{C_{01}^{\mu}(\overline{\Omega})} \\ &\quad \leqslant \frac{M(\mu)}{\alpha(1-\alpha)} \|f\|_{C_{0T}^{2+\alpha,\alpha}(C_{01}^{\mu}(\overline{\Omega}))} + M(\mu) \left[\sum_{r=1}^{n} \left\| a_r(x) \frac{\partial^2 \varphi}{\partial x_r^2} \right\|_{C_{01}^{2+\mu}(\overline{\Omega})} \right. \\ &\quad\quad \left. + \sum_{r=1}^{n} \left\| a_r(x) \frac{\partial^2 \xi}{\partial x_r^2} \right\|_{C_{01}^{2+\mu}(\overline{\Omega})} + \sum_{r=1}^{n} \left\| a_r(x) \frac{\partial^2 \psi}{\partial x_r^2} \right\|_{C_{01}^{2+\mu}(\overline{\Omega})} \right], \end{aligned} \tag{34}$$

where $0 < \alpha < 1$, $\mu = (\mu_1, \dots, \mu_n)$, $0 < \mu_i < 1$, $1 \leqslant i \leqslant n$. $M(\mu)$ *is independent of* α, $\varphi(x)$, $\xi(x)$, $\psi(x)$ *and* $f(t,x)$.

Third, in $[0,T] \times \Omega$, we consider the mixed boundary value problem for the multidimensional elliptic equation

$$\begin{aligned}
&-u_{tt}(t,x) - \sum_{r=1}^{n} a_r(x) \frac{\partial^2 u}{\partial x_r^2} + \delta u(t,x) = f(t,x) + p(x), \quad x \in \Omega,\ 0 < t < T, \\
&u(0,x) = \varphi(x), \quad u(T,x) = \psi(x), \quad u(\lambda,x) = \xi(x), \quad x \in \overline{\Omega}, \\
&\frac{\partial u}{\partial \vec{n}}(t,x) = 0, \quad x \in S.
\end{aligned} \tag{35}$$

The differential expression (33) defines a positive operator A^x acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $\partial u / \partial \vec{n} = 0$ on S. Therefore, we can use results of abstract Theorems 1 and 2 to get the following theorem.

Theorem 6. *For the solution of boundary value problem* (35)*, the coercive estimate* (34) *holds.*

4 Conclusion

In the present paper, the well-posedness of the boundary value abstract elliptic problem with the unknown parameter in Holder spaces with a weight is established. In practice, new Schauder type exact estimates in Holder norms for the solution of three boundary value problems for elliptic equations with the unknown parameter are obtained. Moreover, applying the result of the monograph [29] the high order of accuracy two-step difference schemes for the numerical solution of the boundary value elliptic problem with the unknown parameter can be presented. Of course, the coercive stability estimates for the solution of these difference schemes have been established without any assumptions about the grid steps.

References

1. A. Ashyralyev, On the problem of determining the parameter of a parabolic equation, *Ukr. Math. J.*, **62**(9):1397–1408, 2011.

2. C. Ashyralyyev, High order of accuracy difference schemes for the inverse elliptic problem with Dirichlet condition, *Bound. Value Probl.*, **2014**(5), 23 pp., 2014.

3. C. Ashyralyyev, M. Dedeturk, A finite difference method for the inverse elliptic problem with the Dirichlet condition, *Contemp. Anal. Appl. Math.*, **1**(2):132–155, 2013.

4. C. Ashyralyyev, M. Dedeturk, Approximate solution of inverse problem for elliptic equation with overdetermination, *Abstr. Appl. Anal.*, **2013**, Article ID 548017, 19 pp., 2013.

5. C. Ashyralyyev, O. Demirdag, The difference problem of obtaining the parameter of a parabolic equation, *Abstr. Appl. Anal.*, **2012**, Article ID 603018, 14 pp., 2012.
6. C. Ashyralyyev, A. Dural, Y. Sözen, Finite difference method for the reverse parabolic problem, *Abstr. Appl. Anal.*, **2012**, Article ID 294154, 17 pp., 2012.
7. A. Ashyralyev, A.S. Erdogan, Well-posedness of the inverse problem of a multidimensional parabolic equaton, *Bulletin of Odessa National University. Mathematics and Mechanics*, **15**(18):129–135, 2010.
8. A. Ashyralyev, A.S. Erdogan, On the numerical solution of a parabolic inverse problem with the Dirichlet condition, *Int. J. Math. Comput.*, **11**(J11):73–81, 2011.
9. A. Ashyralyev, A.S. Erdogan, O. Demirdag, On the determination of the right-hand side in a parabolic equation, *Appl. Numer. Math.*, **62**(11):1672–1683, 2012.
10. A. Ashyralyev, M. Urun, Determination of a control parameter for the Schrodinger equation, *Contemp. Anal. Appl. Math.*, **1**(2):156–166, 2013.
11. M. Dehghan, Determination of a control parameter in the two-dimensional diffusion equation, *Appl. Numer. Math.*, **37**(4):489–502, 2001.
12. Y.S. Eidel'man, An inverse problem for an evolution equation, *Math. Notes*, **49**(5):535–540, 1991 (in Russian).
13. D.G. Orlovskii, Inverse Dirichlet problem for an equation of elliptic type, *Differ. Equ.*, **44**(1):124–134, 2008.
14. D. Orlovsky, S. Piskarev, On approximation of inverse problems for abstract elliptic problems, *J. Inverse Ill-Posed Probl.*, **17**(8):765–782, 2009.
15. D. Orlovsky, S. Piskarev, The approximation of Bitzadze–Samarsky type inverse problem for elliptic equations with Neumann conditions, *Contemp. Anal. Appl. Math.*, **1**(2):118–131, 2013.
16. A.I. Prilepko, D.G. Orlovsky, I.A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, New York, 1987.
17. A.A. Samarskii, P.N. Vabishchevich, *Numerical Methods for Solving Inverse Problems of Mathematical Physics*, Inverse Ill-Posed Probl. Ser., Walter de Gruyter, Berlin, New York, 2007.
18. V.V. Soloviev, Inverse problems of source determination for the two-dimensional Poisson equation, *Zh. Vychisl. Mat. Mat. Fiz.*, **44**(5):862–871, 2004 (in Russian).
19. A. Ashyralyev, On well-posedness of the nonlocal boundary for elliptic equations, *Numer. Funct. Anal. Optim.*, **24**(1–2):1–15, 2003.
20. A. Ashyralyev, A note on the Bitsadze–Samarskii type nonlocal boundary value problem in a Banach space, *J. Math. Anal. Appl.*, **344**(1):557–573, 2008.
21. A. Ashyralyev, E. Ozturk, On Bitsadze–Samarskii type nonlocal boundary value problems for elliptic differential and difference equations: Well-posedness, *Appl. Math. Comput.*, **219**(3):1093–1107, 2013.
22. A. Ashyralyev, F.S.O. Tetikoglu, FDM for elliptic equations with Bitsadze–Samarskii–Dirichlet conditions, *Abstr. Appl. Anal.*, **2012**, Article ID 454831, 22 pp., 2012.

23. G. Berikelashvili, N. Khomeriki, On a numerical solution of one nonlocal boundary-value problem with mixed Dirichlet–Neumann conditions, *Lith. Math. J.*, **53**(4):367–380, 2013.

24. V.A. Il'in, E.I. Moiseev, Two-dimensional nonlocal boundary-value problem for the Poisson's operator in differential and difference interpretations, *Mat. Model.*, **2**(8):139–156, 1990 (in Russian).

25. K. Jakubėlienė, R. Čiupaila, M. Sapagovas, Solution of a two-dimensional elliptic equation with a nonlocal condition, in: *Proceedings of the International Conference on Differential Equations and Their Applications (Panevėžys, Lithuania, 10–12 September, 2009)*, Technologija, Kaunas, 2009, pp. 139–146.

26. M.P. Sapagovas, Difference method of increased order of accuracy for the Poisson equation with nonlocal conditions, *Differ. Equ.*, **44**(7):1018–1028, 2008.

27. M.P. Sapagovas, A.D. Štikonas, On the structure of the spectrum of a differential operator with a nonlocal condition, *Differ. Equ.*, **41**(7):961–969, 2005.

28. M. Sapagovas, A. Štikonas, O. Štikonienė, Alternating direction method for the Poisson equation with variable weight coefficients in an integral condition, *Differ. Equ.*, **47**(8):1163–1174, 2011.

29. A. Ashyralyev, P.E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, Operator Theory Advances and Applications, Vol. 148, Birkhäuser, Basel, Boston, Berlin, 2004.

30. A. Ashyralyev, P.E. Sobolevskii, *Well-Posedness of Parabolic Difference Equations*, Operator Theory Advances and Applications, Vol. 69, Birkhäuser, Basel, Boston, Berlin, 1994.