The Use of Symmetric Quartic Potential Well for Noise Filtering

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Nurhan Güneş
Gümüshane Üniversitesi

Mark Leeson
The University of Warwick

The Use of Symmetric Quartic Potential Well for Noise Filtering

Nurhan Gunes* and Mark S. Leeson

School of Engineering, University of Warwick
Coventry, CV4 7AL, United Kingdom
n.gunes@warwick.ac.uk

Matthew D. Higgins

Warwick Manufacturing Group

International Digital Laboratory, University of Warwick
Coventry, CV4 7AL, United Kingdom

In this letter, a symmetric quartic potential well is considered for use as a low-pass filter in BPAM receivers. By using the BPAM input to force a change in the shape of the well, the output can be considered as the position of a particle in the well. The potential well has been designed to pass the BPAM signal and suppress the background noise. Within the paper, DC gain and cut-off frequency are defined where the potential well is subject to a noise free single pulse. The effects of the background noise are also discussed. These initial results are applicable to other pulse modulation methods. Moreover, it is revealed that the cut-off frequency plays a significant role on the BPAM detector. Finally, a new result for the probability of error is introduced which clearly shows that the potential well filters out the background noise as much as a lowpass Butterworth filter does.

1. Introduction

Potential well models are significant to researchers in explaining the reasoning behind probabilistic behaviours. For example, the motion of a particle in a double well potential is used to model the dynamic behaviour of an RF superconducting quantum interference device (SQUID) [1], a finite square potential well is used to model the energy of the electron in a single quantum well [2], and a Morse potential well is used to model the vibrational structure of molecules [3]. Moreover, potential wells can exhibit a behaviour phenomenon called Stochastic Resonance (SR), which occurs when a nuisance signal, such as noise, counter-intuitively, favours the system [4]. This combination of potentially beneficially behavioural phenomenon, coupled with the links to probabilistic concepts analogous in information theory, make them important for further consideration in the communications and signal processing fields.

Of particular interest to researchers, and a subject that has been studied extensively in the literature, is that of the Brownian motion of a particle in a double well. It has been shown that this system can exploit the background noise to improve the SNR [5-8]. To take advantage of the double well potential behaviors, they are effectively implemented as signal processing block sets within the receiver architecture. Examples of implementation have been shown for binary phase shift keying (BPSK) [9], frequency shift keying (FSK) [10], minimum shift keying (MSK) [11], and binary pulse amplitude modulation (BPAM) [12-17]. Upon implementation, these block sets are more generically referred to as a *stochastic resonator.* These studies have all shown that a stochastic resonator, namely a Symmetric Quartic Potential Well (SQPW), exhibiting the SR phenomenon can enhance the system performance.

The SQPW considered here also has a filter like response which thus results in an SNR gain. The frequency response has a pattern similar to a Low Pass Filter (LPF) [18-20], and in some cases, linear response functions can therefore be derived [5, 21]. What can be noted is that of all the papers given so far, none of them list their main focus as designing this SQPW as a filter, focusing instead their efforts upon the SNR gain itself. Thus, whilst they do provide analysis along the lines of SNR, BER, Mutual Information etc., the fundamental filtering characteristics; like DC gain and cutoff frequency, are not presented as a focused contribution.

Therefore, this paper aims to make a significant contribution to field by designing the SQPW as an LPF within a BPAM receiver. Through a process of designing the SQPW to pass a BPAM signal and attenuate the background noise, an analytical definition for the DC gain and cut-off is shown. This result, novel in itself, is then validated through simulation of the SQPW and an output-input magnitude ratio response (MRR). Further to this, the effect of the background noise on the MRR is then investigated. Finally, the primary result of the paper, that of two equations that define both the probability of error and cut-off frequency for the SQPW, are shown. These two final closed form equations are new to the literature.

2. Receiver Model

2.1. *Receiver and Input Signal*

It is assumed that the transmitter sends digital information by use of two rectangular pulses ($g_o(t) = -A, g_1(t) = A$). Each pulse is transmitted within the symbol interval of duration $T_b(0 < t \leq T_b)$. Therefore, the transmitted signal is in the following form

$$s(t) = \sum_{i=1}^{\infty} g_{B_i}(t - [i-1]T_b) \tag{1}$$

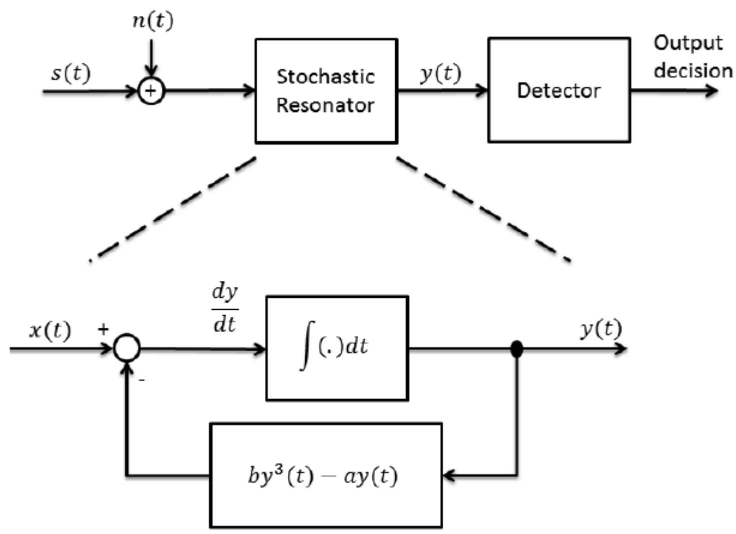

Fig.1. AWGN Channel and Receiver configuration

where B_i is a sequence of binary symbols. Each symbol is an identically distributed and independent random variable with $P(B_i = 1) = P(B_i = 0) = 0.5$.

The channel is assumed to corrupt the signal by the addition of white Gaussian noise, n(t), with autocorrelation function $B_i\langle n(t)n(t+\tau)\rangle = N_0/2\,\delta(t)$. Based on the received signal, the aim is to design a receiver that consists of a SQPW and a detector as shown in Fig. 1.

The function of the detector is to decide which of two pulses was transmitted based on the sample taken at the end of each pulse. Therefore, the output bit sequence can be given by:

$$B_i^{out} = \begin{cases} 0, & y(iT_b) < 0 \\ 1, & y(iT_b) \geq 0 \end{cases} \tag{2}$$

2.2. *SQPW and Filter Design*

As noted, the most studied stochastic resonator is SQPW based upon the Brownian motion in the symmetric quartic bi-stable potential under the action of an external field. Here, the transmitted signal $s(t)$ is considered as the external field. The internal state, which is the output $y(t)$, evolves according to

$$\frac{dy}{dt} = ay - by^3 + s(t) + n(t). \tag{3}$$

It is known that at high frequencies the output power of the SQPW falls off [5,18]. Considering that the input noise has a flat spectrum, this fall-off results in attenuation. Therefore $|Y(f)|/|S(f) + N(f)|$ needs to show this attenuation, which can be defined as an output-input magnitude ratio response (MRR) function $|H(f)|$. Although this notation is used for linear systems, it is found convenient whilst seeking a filter like response.

Assuming that a pulse is applied to the input, and the SQPW is designed to provide a pulse like shape corresponding to the input, then it can be said that the SQPW allows that pulse to occur in its output [16]. As the pulse signal is baseband, this SQPW passes the low (baseband) frequencies. Therefore, the MRR $|H(f)|$ needs to show the same relation.

When a pulse with the amplitude A is applied, then the output will be stable at one of the roots of $ay - by^3 + A = 0$. If $|A|$ is greater than potential barrier $\sqrt{4a^3/27b}$, there is only one real root which has the same sign with input. Defining the output $y = k \times A$, where k can be determined from $bk^3A^2 - ak - 1 = 0$, the DC response of the SQPW becomes:

$$|H(0)| = k \tag{4}$$

The frequency response is a more complex problem, and is related to the timing such that when the output is already stable at $-kA$, a pulse with amplitude $A \gg \sqrt{4a^3/27b}$ and a duration T_b must be able to change the output to kA in the same period. In this case, the output is a monotonically increasing function since it has one real root. Therefore, there is a specific point in time which makes the integral of output zero. For example, the output reaches ckA at $t = T$ where $0 < c \leq 1$, the integral of (3) from $t = 0$ to T can be given by

$$ckA - (-kA) = \int_0^T (ay - by^3)dt + \int_0^T Adt, \tag{5}$$

where T is the specific time. In order to simplify (5), the output is assumed anti-symmetric, which makes the integral of y^3 zero. Thus, it can be written that $1 < T/k \leq 2$, which shows that $T = 2k$ is the critical time for the output to be stable at the end of each pulse. Therefore, the SQPW passes the transmitted signal if $T_b \geq 2k$, which means that the cut-off frequency of the system is

$$f_c \leq 1/2k. \tag{6}$$

3. Simulation Method and MRR

The design above can be verified by a simulation. This requires a specific treatment, known as stochastic differential equations (SDEs), and needs to be considered in discrete time.

3.1. *Heun Scheme*

The output of the chosen SQPW evolves according to (3) which is a one dimensional SDE and can be solved by SDE integration schemes.

Firstly, it has to be written in the generic form that is given by $x = f(x) + g(x)\xi(t)$, where $\boldsymbol{x}$ represents y, $f(x) = ay - by^3 + s(t)$, $g(x) = \sqrt{N_0/2}$, and $\xi(t) = N(0,1)$. Considering the discretization, the formal integration with the step size *h* is given by

$$x(h) - x(0) = \int_0^h \{f(x) + g(x)\xi(t)\}dt, \tag{7}$$

Secondly, a method to solve this integral has to be chosen. In [22], SDE integration schemes are provided with their accuracies. Among those methods, the Heun scheme is considered since it promises a better accuracy and needs less computational effort. It is applied in two steps as,

$$x_1 = x(0) + \sqrt{\frac{N_0}{2}}Z(h) + hf(x[0]) \tag{8}$$

$$x(h) = x(0) + \sqrt{\frac{N_0}{2}}Z(h) + \frac{h}{2}\{f(x_1) + f(x[0])\} \tag{9}$$

where $Z(h)$ is a Gaussian random variable with zero mean and a variance which is equal to the step size h [22].

3.2. *MRR without Noise*

To verify the design, a pulse without the noise term is applied to make the output change from $-kA$ to kA, then the frequency response is obtained from $|H(f)| = |Y(f)|/|S(f)|$ by calculating the DFTs of both input and output.

As it is desired for the pulse to occur at the output, hereafter the potential barrier is eliminated by $a = 0$ and consequently there is no more restriction upon A. The fact is that increasing $|A|$ leads a to lose its effect on the output, and the minimum width of the well, that dominates the maximum output values, depends on b. Additionally, k is set to T_b/m where $m > 2$. Then, the simulation is carried out for different m parameters, which shows how the frequency response evolves.

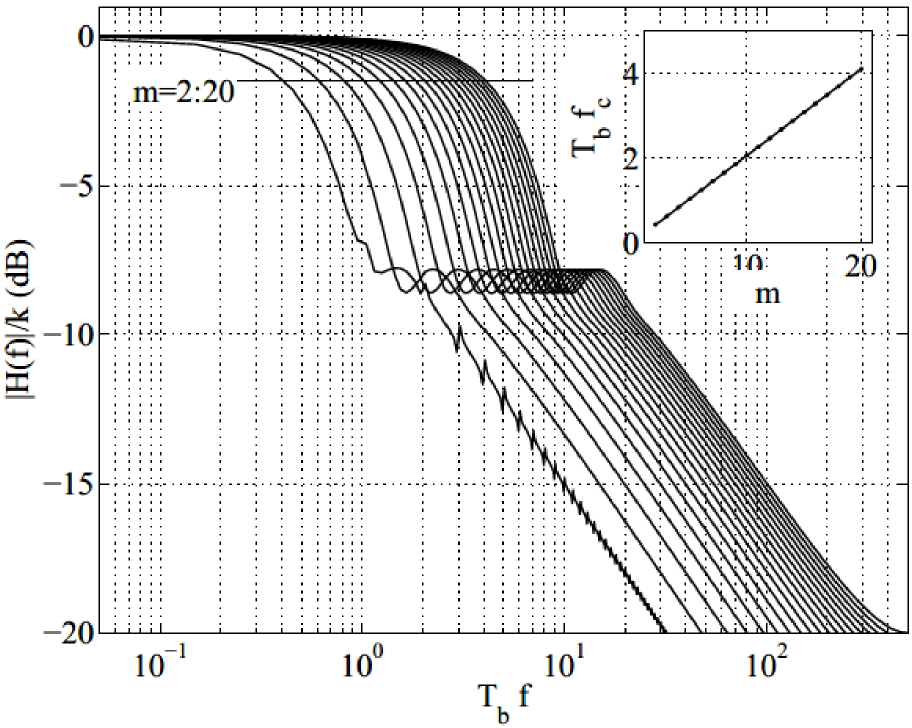

Fig. 2. Normalised frequency response $|H(f)|/k$ where $a = 0$, $b = 1/(k^3A^2)$, $k = T_b/m$, $m = 2:20$ (from left to right). The input is $A = 1$, $T_b = 1$, and $t_s = T_b/10^3$. The insert gives the normalised cut-off frequencies of each curve.

The results are illustrated in Fig. 2. Defining $|H(f_c)|/k = 1/\sqrt{2}$ where f_c is the cut-off frequency, it is found that m shifts $|H(f_c)|/k$ and consequently the cut-off frequency is well-fitted by the expression

$$f_c \approx \frac{0.2047m}{T_b} \approx \frac{1}{5k} \tag{10}$$

Here, (10) verifies via simulation that (6) is correct. Defining the bandwidth from DC to the first null in the frequency spectrum, $m = T_b/k \geq 5$ keeps the signal bandwidth in the pass-band.

By simulation, the design has been verified. It is found that the DC gain is $|H(0)| = k$ as determined previously where $T_b > 2k$. As can be seen, this is equal to (4), thus verifying its correctness. Moreover, decreasing k shifts the $|H(f)|$ curve towards higher frequencies.

3.3. *MRR with Noise*

The MRR is dependent on not only the transmitted signal but also the noise term. Although the noise term was not taken into account previously, those features of the MRR must be considered as references; when the noise term is considered the MRR changes accordingly.

To define the input noise power, the SNR per bit is used as the noise has an infinite number of frequency components. It is denoted by γ_b, and given by $\mathcal{E}_b/N_0$ where $\mathcal{E}_b$ is the pulse energy and equals to A^2T_b for a BPAM signal.

Then, in order to observe the effect of noise, the SQPW is simulated. The results are illustrated in Fig. 3. It shows that the slope in the attenuation band stays almost the same and the noise can decrease the DC gain considerably.

The noise plays a significant role on f_c. Although the slope does not change, DC gain varies depending on γ_b. By definition, f_c is determined from $|H(f_c)| = |H(0)|/\sqrt{2}$. Thus, the cut-off frequency can be derived from

$$\frac{10\log[\phi(\gamma_b)] - 10\log[|H(0)|/k]}{\log[T_b f_c^N] - \log[T_b f_c^O]} \approx -10 \text{ dB (decade)}^{-1} \tag{11}$$

where $\phi(\gamma_b)$ is a function of γ_b and f_c^N and f_c^O are the cut-off frequencies with and without noise respectively. Therefore, f_c can be written in a general form as

$$f_c = \frac{1}{ck\phi(\gamma_b)} \tag{12}$$

where c is a coefficient. Assuming that $\phi(\gamma_b)$ is always positive and it tends towards 0 as γ_b decreases, the noise increases f_c.

In concluding this key result, it implies that at high SNRs, the noise slightly decreases f_c, whilst, at low SNRs, it decreases the DC gain and increases the cut-off frequency.

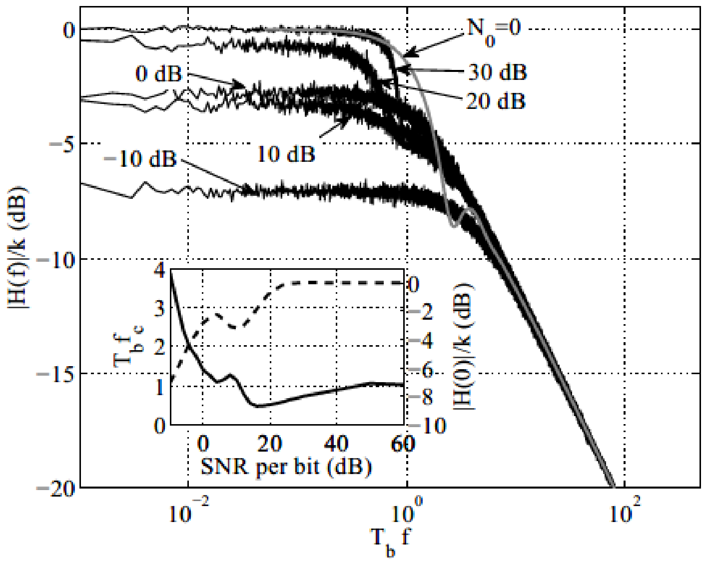

Fig. 3. Normalised frequency response $|H(f)|/k$ where $a = 0$, $b = 1/(k^3A^2)$, $k = T_b/m$, $m = 5$, $A = 1$, $T_b = 1$, $\gamma_b = -10 : 10 : 30$ dB, and $t_s = T_b/10^3$. The insert illustrates the normalised cut-off frequencies (solid) and DC gain (dashed).

4. BER Performance of Receiver

4.1. *The Effect of Cutoff Frequency*

Besides the MRR, the error performance is also needed to validate the filtering capability of the SQPW. If the noise is filtered out, the output SNR must be less than that of the input. As the output SNR is generally used to obtain the probability of error, the BER of the SQPW can provide the SNR at the output. In order to reveal the BER performance of the receiver, it is simulated under various values of γ_b.

Assuming that transmitted signal power stays in the $1/T_b$ bandwidth, for ideal case, the output SNR would be $\gamma_b T_b$ and increasing the cut-off frequency only makes more of the noise component pass. If so, for the receiver, the output SNR can be given by $SNR_O = \gamma_b / f_c$, where $f_c > 1/T_b$.

To verify this relation between SNR_O and γ_b, the BER performance of the receiver is obtained by means of simulation. For each γ_b, simulations continue until the 100^{th} error occurs. The results are illustrated in Fig. 4. From left to right, m, and consequently f_c, increases.

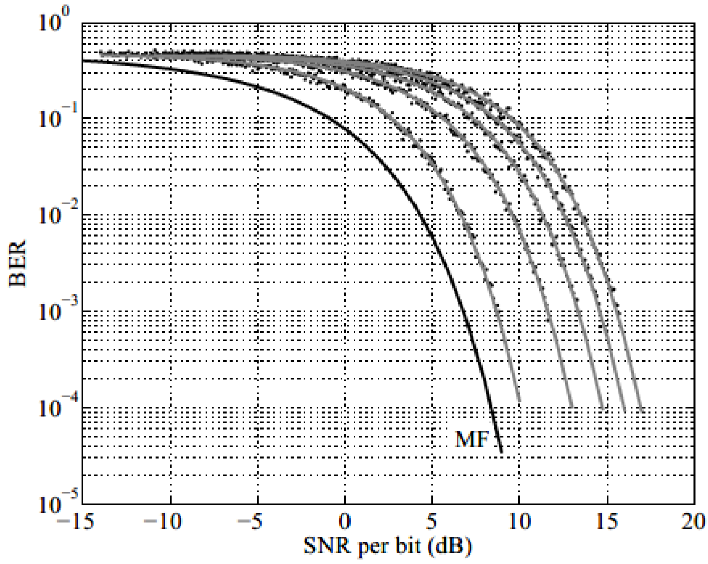

Fig. 4. BER performance of the receiver where $a = 0$, $b = 1/(k^3 A^2)$, $k = T_b/m$, $m = 4 : 4 : 20$ (from left to right), $A = 1$, $T_b = 1$, and $t_s = T_b/10^3$. The black curve with 'MF' mark illustrates BER performance of the receiver with the matched filter instead of the SQPW.

The BER curves follow the same trend, shifted in m. This change in BER, corresponding to f_c given by (12), becomes more noticeable when curve fitting is applied. It is found that the BER curves are well-fitted by

$$P_e = \frac{1}{2}\exp\left\{-\frac{\gamma_b}{c_1 m + c_2} - \frac{1}{c_3}\right\} \tag{13}$$

where $c_1 = 0.2911$, $c_2 = 0.0424$ and $c_3 = 16.2$ with an RMSE = 0.06861 for the argument of the exponential. Fitted curves are also illustrated in Fig. 4. Considering that m > 2 and *yb* >> cim/c_3, (13) can be rewritten as

$$P_e = \frac{1}{2}\exp\left\{-\frac{\gamma_b}{c_f T_b f_c}\right\} \tag{13}$$

where c_f, the filter coefficient, is about 0.2911/0.2047 = 1.422. Note that, in general, the BER is a function of the output SNR in the form of mean over deviation. Therefore, $\gamma_b/(c_f T_b f_c)$ is directly related to the output SNR of the SQPW, which proves that the input signal frequency components below f_c pass. In addition to that, (14) is a supplement to the equation in [15]; the former is specific for the filter usage, while the latter is for the weak signal detection.

4.2. *BER Comparison with Butterworth LPF*

It is shown that there is a directly proportional relation between the output SNR and cutoff frequency. Now, it is desired to compare the SQPW with a conventional filter.

Considering Fig. 2 and Fig. 3, $(|H(f)|/k)^2$ has no ripples in the pass band, and in the stop band there is a -20 dB per decade slope like a first order Butterworth filter.

However, the SQPW's stop band consists of two slopes; the first starts from the -3 dB point, applies to a short frequency range, varies with SNR, and is steeper than the second slope that is fixed at -20 dB per decade. This is an advantage of the SQPW at higher SNRs compared to the Butterworth filter having a cutoff frequency at -3 dB regardless of the filter order. On the other hand, the SNR dependency of its cutoff frequency becomes destructive at lower SNRs.

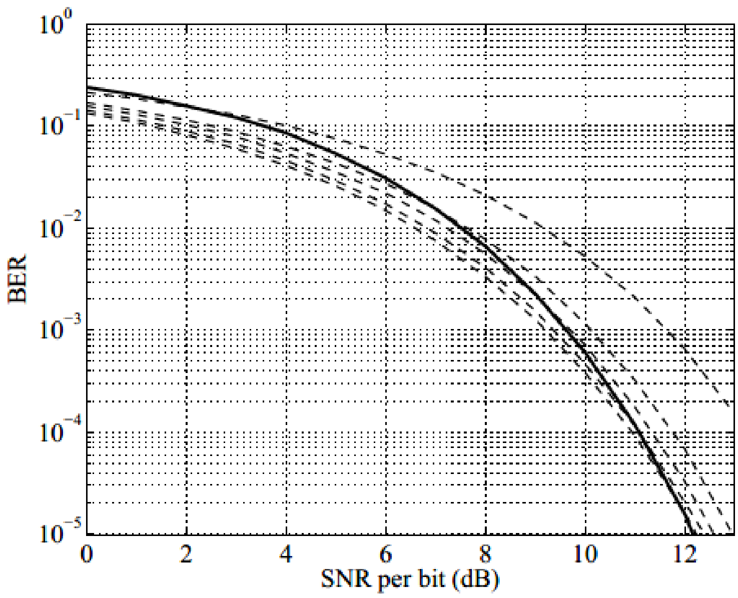

Fig. 5. BER performances of the receiver accommodating either a SQPW (solid) or a Butterworth filter(dashed). The SQPW is with $a = 0$, $b = 1/(k^3A^2)$, $k = T_b/m$, $m = 5$, $A = 1$, $T_b = 1$, and lowpass Butterworth filters are with cutoff frequency $1/T_b$, and the orders 1 : 5 (from right to left). The time step is $t_s = T_b/10^2$.

The effect of this SNR dependency can be observed from the BER performance of the receiver accommodating a SQPW and a low pass Butterworth filter separately. The results are illustrated in Fig. 5 and verify that, at SNRs lower than 2 dB, since a 1st order Butterworth has a fixed cut-off frequency, fewer errors occur in the output, whilst the SQPW provides an error performance as good as a fifth order Butterworth at SNRs higher than 12 dB.

5. Conclusions

In this paper, a SQPW was designed as a low pass filter for use in BPAM receivers. The DC gain and cut-off frequency were defined and verified by SDE simulation where the potential barrier was not present and where the SQPW was subject to only one pulse. Then, the effect of noise was examined and discussed. Following that, an expression for the error performance of a BPAM receiver with a SQPW was obtained and compared with one having the Butterworth filter instead, which also validates the filtering capability of the SQPW.

As the SQPW is a nonlinear system, these filter characteristics vary with input signals. In addition to that, the SR phenomenon was not observed, consequently the SQPW performance lags behind the matched filter because of having no potential barrier. However, such a SQPW becomes imperative if traditional ones cannot be realised, even though its nonlinearity and performance are not favorable when compared to the traditional signal processing methods.

As a result, the DC gain and cutoff frequency presented in this work provide a firm base for studies on SQPW with pulse modulation methods. The use of SQPW as a filter is revealed for the cases where the input dominates the potential barrier, so that the design provides an overall noise suppression

The possible further directions can be: investigating the effect of barrier on the MRR and connecting that effect with SR phenomenon, comparing the performances of different pulse modulation methods, and seeking an alternative decision making strategy.

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