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Research Article

Finite Difference Method for the Reverse Parabolic Problem

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A finite difference method for the approximate solution of the reverse multidimensional parabolic differential equation with a multipoint boundary condition and Dirichlet condition is applied. Stability, almost coercive stability, and coercive stability estimates for the solution of the first and second orders of accuracy difference schemes are obtained. The theoretical statements are supported by the numerical example.

1. Introduction

In the study of boundary value problems for partial differential equations, the role played by the well-posedness (coercivity inequalities) is well known (see, e.g., [1–3]). Well-posedness of nonlocal boundary value problems for partial differential equations of parabolic type has been studied extensively by many researchers (see, e.g., [4–15] and the references therein).

In the paper [4], Ashyralyev studied the positivity of second-order differential and difference operators with nonlocal condition and the structure of interpolation spaces generated by these operators in a Banach space. Applying this result, he obtained the coercive inequalities for the solutions of the nonlocal boundary value problem for differential and difference equations.

In [5], Ashyralyev et al. considered the nonlocal boundary value problem

$$v'(t) + Av(t) = f(t), \quad 0 < t < 1, \ v(0) = v(\lambda) + \mu, \ 0 < \lambda \leq 1, \tag{1.1}$$

in a Banach space with strongly positive operator A. They established the well-posedness of problem (1.1) in Hölder spaces. Moreover, they obtained the exact Schauder's estimates in Hölder norms of solutions of the boundary values problem for $2m$-th order multidimensional parabolic equations.

Ashyralyev established in [6] the well-posedness of the nonlocal boundary-value problem (1.1) in Bochner spaces. He considered the first and second order of accuracy difference schemes for the approximate solutions of problem (1.1). He also established the coercive inequalities for the solutions of these difference schemes. Moreover, in applications, he obtained the almost coercive stability and coercive stability estimates for the solutions of difference schemes for the approximate solutions of the nonlocal boundary-value problem for parabolic equation.

Clément and Guerre-Delabriére studied in [8] maximal regularity (in the L_p-sense) for abstract Cauchy problems of order one and boundary value problems of order two. As is well-known regularity of the first problems implies regularity of the second ones; they also proved that the converse to hold if the underlying Banach space has the UMD property. A stronger notion of regularity, which is introduced by Sobolevskii, plays an important role in the proofs.

In [9], Gulin et al. considered the linear heat equation:

$$u_t = u_{xx} + u_{yy}, \quad 0 < x < 1, \; 0 < y < 1, \tag{1.2}$$

with Dirichlet condition $u(x, 0, t) = u(x, 1, t) = 0$, $0 \le x \le 1$ and nonlocal boundary conditions $u(0, y, t) = 0$, $u_x(0, y, t) = u_x(1, y, t)$, $0 \le y \le 1$. They constructed an explicit difference scheme with second order of approximation with respect to the space variables and first order of approximation with respect to t. Moreover, using previous results of Ionkin and Morozova for the one-dimensional heat equation with nonlocal boundary conditions, they proved the stability of this scheme with respect to the D_1-norm $\|y\|_{D_1} = (D_1 y, y)^{1/2}$, which is induced by the symmetric and positive-definite matrix D_1.

Liu et al. studied in [10] a finite difference method for multidimensional nonlinear coupled system of parabolic and hyperbolic equations. By using a variational method, they obtained an a priori estimate. They also proved that the finite difference scheme is uniquely solvable and unconditionally stable. To support the theory, they gave numerical example of two-dimensional problem.

In [11, 12], Martin-Vaquero and Vigo-Aguiar provided algorithms improving the CPU time and accuracy of Crandall's formula. They studied the convergence of the algorithms and compared the efficiency of the methods with well-known numerical examples.

In [13], Sapagovas applied finite difference approximations to a nonhomogeneous heat equation in one space dimension, subject to nonlocal boundary conditions. He presented a stable difference approximation for the equation and a piecewise constant discretization of the integrals appearing in the boundary conditions. He discussed the stability of the complete problem with respect to two parameters included in the integral terms. He constructed a stability region in the plane of the parameters and gave practical examples with specific choices of the integral conditions. Sapagovas investigated in [14] the stability of implicit difference schemes for the equation of a thermoelastic rod, which is a parabolic equation subject to integral conditions for the boundaries.

In [15], Shakhmurov dealt with a nonlocal boundary value problem for a degenerate equation in a Banach space E with unbounded operators in E. He proved the maximal L_p

regularity and Fredholmness of the problem. He also applied the results to nonlocal boundary value problems for degenerate elliptic and quasielliptic differential equations and their finite or infinite systems on cylindrical domains.

It is well known that reverse problems arise in various applications, for example, boundary layer problems in fluid dynamics [16, 17], plasma physics, and astrophysics in the study of propagation of an electron beam through the solar corona [18]. For further applications of such problems, we refer the reader to [19–22] and the references therein.

In the paper [23], Ashyralyev et al. considered the multipoint nonlocal boundary value problem for reverse parabolic equations

$$\frac{du(t)}{dt} - Au(t) = f(t), \quad (0 \leq t \leq 1),$$
$$u(1) = \sum_{k=1}^{p} \alpha_k u(\theta_k) + \varphi, \tag{1.3}$$
$$0 \leq \theta_1 < \theta_2 < \cdots < \theta_p < 1$$

in a Hilbert space H with self-adjoint positive definite operator A.

$u(t)$ is called a *solution* of problem (1.3) if the following conditions hold:

(1) $u(t)$ is continuously differentiable on the segment $[0,1]$. The derivatives at the end points of the segment are understood as the appropriate unilateral derivatives.

(2) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$ and the function $Au(t)$ is continuous on the segment $[0,1]$.

(3) $u(t)$ satisfies the equation and the nonlocal boundary conditions (1.3).

A solution of problem (1.3) defined in this manner will be from now referred to as a solution of problem (1.3) in the space $C(H) = C([0,1], H)$ of all continuous functions $\varphi(t)$ defined on $[0,1]$ with values in H equipped with the norm

$$\|\varphi\|_{C(H)} = \max_{0\leq t\leq 1} \|\varphi(t)\|_H. \tag{1.4}$$

Problem (1.3) is *well posed* in $C(H)$, if for the solutions of (1.3), we have the following *coercivity inequality:*

$$\|u'\|_{C(H)} + \|Au(t)\|_{C(H)} \leq M\Big(\|f\|_{C(H)} + \|A\varphi\|_H\Big). \tag{1.5}$$

Here, $1 \leq M$ is independent of $f(t) \in C(H)$, $\varphi \in D(A)$.

Throughout the paper, M indicates positive constants which can be different from time to time and we are not interested to make precise. We write $M(\alpha, \beta, \ldots)$ to stress the fact that the constant depends only on $\alpha, \beta, \ldots$

Under the assumption:

$$\sum_{k=1}^{p} |\alpha_k| \leq 1. \tag{1.6}$$

Ashyralyev et al. established in [23] the well-posedness of these problems in the space of smooth functions. In applications, they obtained coercivity estimates for the solution of parabolic differential equations.

Moreover, in [24], Ashyralyev et al. considered the first order of accuracy Rothe difference scheme:

$$\begin{gathered}\tau^{-1}(u_k - u_{k-1}) - Au_{k-1} = \varphi_k, \quad \varphi_k = f(t_k), \\ t_k = k\tau, \quad 1 \le k \le N, \ N\tau = 1, \\ u_N = \sum_{m=1}^{p} \alpha_m u_{\ell_m} + \varphi, \\ \ell_m = \left[\frac{\theta_m}{\tau}\right], \quad 1 \le m \le p,\end{gathered} \tag{1.7}$$

for approximately solving problem (1.3). They established some stability estimates and almost coercivity of the solution for the difference scheme.

In the present paper, multipoint nonlocal boundary value problem for the multidimensional parabolic equation with Dirichlet condition,

$$\begin{gathered}u_t(t,x) + \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = f(t,x), \\ x = (x_1, \ldots, x_n) \in \Omega, \quad 0 < t < 1, \\ u(1,x) = \sum_{i=1}^{p} \alpha_i u(\theta_i, x) + \varphi(x), \quad x \in \overline{\Omega}, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1, \\ u(t,x) = 0, \quad x \in S, \ 0 \le t \le 1\end{gathered} \tag{1.8}$$

under the condition (1.6) is considered. Here, $a_r(x), (x \in \Omega)$, $\varphi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions and $a_r(x) \ge a > 0$, and $\Omega = (0,\ell) \times \cdots \times (0,\ell)$ is the open cube in the n-dimensional Euclidean space with boundary S, $\overline{\Omega} = \Omega \cup S$. In the Hilbert space $H = L_2(\overline{\Omega})$, we introduce the self-adjoint positive definite operator A defined by

$$Au(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r}(x))_{x_r}, \tag{1.9}$$

with domain

$$D(A) = \left\{u, u_{x_r}, u_{x_r x_r} \in L_2\left(\overline{\Omega}\right),\ 1 \le r \le n : u(x)|_{x \in S} = 0\right\}. \tag{1.10}$$

Then, problem (1.8) can be written in the abstract form as the nonlocal boundary value problem for reverse parabolic equation (1.3).

The first and second orders of accuracy in t and the second order of accuracy in space variables for the approximate solution of problem (1.8) are presented. Applying the method of papers [23, 24], the stability, almost coercive stability, and coercive stability estimates for the solution of these difference schemes are obtained. The modified Gauss elimination method for solving these difference schemes in the case of one-dimensional parabolic partial differential equations is used.

2. Difference Schemes: Stability Estimates

We will discretize problem (1.8) in two steps. In the first step, we define the grid spaces

$$\begin{gathered}\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \ldots, h_n m_n); m = (m_1, \ldots, m_n), m_r = 0, \ldots, N_r, h_r N_r = 1, r = 1, \ldots, n\}, \\ \Omega_h = \widetilde{\Omega}_h \cap \Omega, \qquad S_h = \widetilde{\Omega}_h \cap S.\end{gathered} \tag{2.1}$$

We denote that $|h| = \sqrt{h_1^2 + \cdots + h_1^2}$. Let $L_{2h} = L_2(\widetilde{\Omega}_h)$ denote the Banach space of grid functions:

$$\varphi^h(x) = \{\varphi(h_1 m_1, \ldots, h_n m_n)\}, \tag{2.2}$$

defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2}. \tag{2.3}$$

To the differential operator A generated by problem (1.8), we assign the second-order approximation difference operator $A_h^x = C_h^x + B_h^x$ acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$. Assume that C_h^x is self-adjoint, positive-definite operator in L_{2h} and $(C_h^x)^{-1} B_h^x$ is bounded operator in L_{2h}.

By using A_h^x, we arrive at the multipoint nonlocal boundary value problem:

$$\begin{gathered}\frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), \quad 0 < t < 1, \ x \in \widetilde{\Omega}_h, \\ u^h(1,x) = \sum_{m=1}^{p} \alpha_m u^h(\theta_m, x) + \varphi^h(x), \quad x \in \widetilde{\Omega}_h,\end{gathered} \tag{2.4}$$

for a finite system of ordinary differential equations with a fixed $|h|$. Note that $|h| = \sqrt{h_1^2 + \cdots + h_n^2} \to 0$. Therefore, we will try to obtain stability, coercivity stability, and almost coercivity estimates with constants independent of $|h|$.

In the second step, problem (2.4) is replaced by the first order of accuracy difference scheme

$$\begin{aligned}
&\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}-A_h^x u_{k-1}^h(x)=f_k^h(x),\\
&f_k^h(x)=f^h(t_k,x),\quad t_k=k\tau,\ 1\le k\le N,\ x\in\widetilde{\Omega}_h,\\
&u_N^h(x)=\sum_{m=1}^{p}\alpha_m u_{\ell_m}^h(x)+\varphi^h(x),\quad x\in\widetilde{\Omega}_h,\\
&\ell_m=\left[\frac{\theta_m}{\tau}\right],\quad m=1,\dots,p,
\end{aligned}\tag{2.5}$$

and the second order of accuracy difference scheme

$$\begin{aligned}
&\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}-A_h^x B_h^x u_{k-1}^h(x)=f_k^h(x),\\
&B_h^x=I+\frac{\tau A_h^x}{2},\quad f_k^h(x)=B_h^x f^h\left(t_{k-(\tau/2)},x\right),\\
&t_k=k\tau,\quad 1\le k\le N,\ N\tau=1,\ x\in\widetilde{\Omega}_h,\\
&u_N^h(x)=\sum_{m=1}^{p}\alpha_m\left\{\left(I+d_mA_h^x\right)u_{\ell_m}^h(x)+d_mB_h^x\varphi_{\ell_m+1}\right\}+\varphi^h(x),\quad x\in\widetilde{\Omega}_h,\\
&d_m=\theta_m-\left[\frac{\theta_m}{\tau}\right]\tau,\quad \ell_m=\left[\frac{\theta_m}{\tau}\right],\quad m=1,\dots,p,
\end{aligned}\tag{2.6}$$

where $[\cdot]$ denotes the greatest integer function.

To formulate our results, let $L_{2h}=L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2=W_2^2(\widetilde{\Omega}_h)$ be spaces of the grid functions $\varphi^h(x)=\{\varphi(h_1m_1,\dots,h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\begin{aligned}
&\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{x\in\widetilde{\Omega}_h}\left|\varphi^h(x)\right|^2h_1\cdots h_n\right)^{1/2},\\
&\left\|\varphi^h\right\|_{W_{2h}^2}=\left\|\varphi^h\right\|_{L_{2h}}+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n}\left|\left(\varphi^h\right)_{x_r}\right|^2h_1\cdots h_n\right)^{1/2}\\
&\qquad+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n}\left|\left(\varphi^h(x)\right)_{x_r\overline{x_r},m_r}\right|^2h_1\cdots h_n\right)^{1/2}.
\end{aligned}\tag{2.7}$$

Furthermore, let $[0,1]_\tau=\{t_k=k\tau,1\le k\le N,\ N\tau=1\}$ be the *uniform grid space* with step size $\tau>0$, where N is a fixed positive integer. We denote $\mathcal{F}_\tau(H)=\mathcal{F}([0,1]_\tau,H)$ for the linear space of grid functions $\varphi^\tau=\{\varphi_k\}_1^N$ with values in the Hilbert space H.

For $\alpha \in [0,1]$, let $C^{\alpha}(H) = C^{\alpha}([0,1]_{\tau}, H)$ and $C_1^{\alpha}(H) = C_1^{\alpha}([0,1]_{\tau}, H)$ be, respectively, the Hölder space and the weighted Hölder space with the norms

$$\begin{aligned} \|\varphi^{\tau}\|_{C^{\alpha}(H)} &= \|\varphi^{\tau}\|_{C_{\tau}(H)} + \max_{1\le k<k+r\le N} \frac{\|\varphi_{k+r}-\varphi_k\|_H}{(r\tau)^{\alpha}}, \\ \|\varphi^{\tau}\|_{C_1^{\alpha}(H)} &= \|\varphi^{\tau}\|_{C_{\tau}(H)} + \max_{1\le k<k+r\le N} \frac{((N-k)\tau)^{\alpha}\|\varphi_{k+r}-\varphi_k\|_H}{(r\tau)^{\alpha}}. \end{aligned} \tag{2.8}$$

Here, $C_{\tau}(H) = C([0,1]_{\tau}, H)$ is the Banach space of bounded grid functions with norm:

$$\|\varphi^{\tau}\|_{C_{\tau}(H)} = \max_{1\le k\le N} \|\varphi_k\|_H. \tag{2.9}$$

Theorem 2.1. *Let τ and $|h|$ be sufficiently small positive numbers. Then, for the solutions of difference schemes* (2.5) *and* (2.6), *the following stability estimate holds:*

$$\left\|\left\{u_k^h\right\}_0^N\right\|_{C_{\tau}(L_{2h})} \le M(\delta,\theta_p)\left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\left\{f_k^h\right\}_1^N\right\|_{C_{\tau}(L_{2h})}\right], \tag{2.10}$$

where $M(\delta,\theta_p)$ is independent of τ, h, $\varphi^h(x)$, and $f_k^h(x)$ and $k = 1,\dots,N$.

Proof. The proof of Theorem 2.1 is based on the formulas for the solution of difference scheme (2.5)

$$u_k^h = R^{N-k}u_N^h - \sum_{j=k+1}^{N} R^{j-k} f_j^h \tau, \quad 0 \le k \le N-1, \tag{2.11}$$

$$u_N^h = T_{\tau}\left(-\sum_{k=1}^{p}\sum_{j=\ell_k+1}^{N} \alpha_k R^{j-\ell_k} f_j^h \tau + \varphi^h\right), \tag{2.12}$$

and for the solution of difference scheme (2.6)

$$u_k^h = D^{N-k}u_N^h - \sum_{j=k+1}^{N} D^{j-k} f_j^h \tau, \quad 0 \le k \le N-1, \tag{2.13}$$

$$u_N^h = T_{\tau}'\left\{-\sum_{m=1}^{p}\sum_{j=\ell_m+1}^{N} \alpha_m\left(I + d_m A_h^x\right) D^{j-\ell_m} f_j^h \tau + \sum_{m=1}^{p} \alpha_m d_m \left(B_h^x\right)^{-1} f_{\ell_m+1}^h + \varphi^h\right\}. \tag{2.14}$$

Here,

$$R=\left(I+\tau A_h^x\right)^{-1},\qquad D=\left(I+\tau A_h^x+\frac{\left(\tau A_h^x\right)^2}{2}\right)^{-1},$$
$$T_\tau=\left(I-\sum_{k=1}^{p}\alpha_k R^{N-[\theta_k/\tau]}\right)^{-1},\qquad T'_\tau=\left(I-\sum_{k=1}^{p}\alpha_k\left(I+d_kA_h^x\right)D^{N-[\theta_k/\tau]}\right)^{-1}. \tag{2.15}$$

By the spectral representation of self-adjoint positive definite operator and the triangle inequality, we have

$$\|T_\tau\|_{H\to H}\le\sup_{\delta\le\mu}\frac{1}{\left|1-\sum_{k=1}^{p}\alpha_k\left(1+\tau\mu\right)^{-N+[\theta_p/\tau]}\right|}\le M(\delta,\theta_p). \tag{2.16}$$

Similarly, we have

$$\left\|T'_\tau\right\|_{H\to H}\le M(\delta,\theta_p). \tag{2.17}$$

Estimates (2.16) and (2.17) conclude the proof of Theorem 2.1. □

Theorem 2.2. *Let τ and $|h|$ be sufficiently small positive numbers. Then, for the solutions of difference problem* (2.5) *and* (2.6), *the following almost coercivity inequality*

$$\left\|\left\{\tau^{-1}\left(u_k^h-u_{k-1}^k\right)\right\}_1^N\right\|_{C_\tau(L_{2h})}\le M(\delta,\theta_p)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\ln\frac{1}{\tau+|h|}\right] \tag{2.18}$$

is valid, where $M(\delta,\theta_p)$ does not depend on $\tau,\ h,\ \varphi^h(x), f_k^h(x),\ k=1,\dots,N$.

Proof. Using formulas (2.11)–(2.14), estimates (2.16) and (2.17), the triangle inequality, assumption (1.6), we obtain

$$\begin{aligned}&\left\|\left\{\tau^{-1}\left(u_k^h-u_{k-1}^k\right)\right\}_1^N\right\|_{C_\tau(L_{2h})}\\&\quad\le M(\delta,\theta_p)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\min\left\{\ln\frac{1}{\tau},1+\left|\ln\left\|A_x^h\right\|_{L_{2h}\to L_{2h}}\right|\right\}\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\right].\end{aligned} \tag{2.19}$$

Since

$$\left|\ln\left\|A_x^h\right\|_{L_{2h}\to L_{2h}}\right|\le M\ \ln\frac{1}{|h|}, \tag{2.20}$$

we have that

$$\min\left\{\ln\frac{1}{\tau}, 1+\left|\ln\left\|A_x^h\right\|_{L_{2h}\to L_{2h}}\right|\right\} \leq M_1 \ \ln\frac{1}{\tau+|h|}. \tag{2.21}$$

From that, inequality (2.19), and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h} it follows inequality (2.18). Theorem 2.2 is proved. □

Theorem 2.3 (see [25, 26]). *For the solution of the elliptic difference problem:*

$$\begin{aligned} A_h^x u^h(x) &= \omega^h(x), \quad x \in \widetilde{\Omega}_h, \\ u^h(x) &= 0, \quad x \in S_h, \end{aligned} \tag{2.22}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n}\left\|\left(u_k^h\right)_{\overline{x}_r\overline{x}_r,j_r}\right\|_{L_{2h}} \leq M\left\|\omega^h\right\|_{L_{2h}}, \tag{2.23}$$

where M does not depend on h and ω^h.

Theorem 2.4. *Let τ and $|h|$ be sufficiently small positive numbers. Then, the solutions of difference problem* (2.5) *and* (2.6) *satisfy the following coercivity stability estimate:*

$$\left\|\left\{\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\}_1^N\right\|_{C_1^\alpha(L_{2h})} \leq M(\delta,\theta_p,\alpha)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_1^\alpha(L_{2h})}\right], \tag{2.24}$$

where $M(\delta,\theta_p,\alpha)$ is independent of $\tau,\ h,\ f_k^h(x)$, and $\varphi^h(x),\ k=1,\ldots,N$.

Theorem 2.5. *Let $A_h^x\varphi^h(x) = f_N^h(x) - \sum_{k=1}^{p}\alpha_m f_{\ell_m}^h(x)$. Then, for solutions of problem* (2.5) *and* (2.6)*, the following coercive stability estimate holds:*

$$\left\|\left\{\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\}_1^N\right\|_{C^\alpha(L_{2h})} + \left\|\left\{u_k^h\right\}_1^N\right\|_{C^\alpha(W_{2h}^2)} \leq M(\delta,\theta_p,\alpha)\left\|\left\{f_k^h\right\}_1^N\right\|_{C^\alpha(L_{2h})}, \tag{2.25}$$

where $M(\delta,\theta_p,\alpha)$ does not depend on $\tau,\ h,\ f_k^h(x)$, and $\varphi^h(x),\ k=1,\ldots,N$.

The proofs of Theorems 2.4–2.5 are based on the formulas:

$$A_h^x u_{k-1}^h = R^{N-k+1} A_h^x u_N^h - \sum_{j=k+1}^{N} \tau A_h^x R^{j-k+1} \left(f_j^h - f_k^h\right) + \left(R^{N-k+1} - I\right) f_k^h,$$
$$A_h^x B_h^x u_{k-1}^h = D^{N-k+1} A_h^x B_h^x u_N^h - \sum_{j=k+1}^{N} \tau A_h^x \left(I + \frac{\tau A_h^x}{2}\right) D^{j-k+1} \left(f_j^h - f_k^h\right) + \left(D^{N-k+1} - I\right) f_k^h, \tag{2.26}$$

the self-adjoint positive definiteness of the operator A_h^x in L_{2h}, estimates (2.16) and (2.17), the triangle inequality, and assumption (1.6).

3. Numerical Results

For the numerical result, we consider the nonlocal boundary value problem:

$$\frac{\partial u(t,x)}{\partial t} + (2+\cos(x))\frac{\partial^2 u(t,x)}{\partial x^2} - \sin(x)\frac{\partial u(t,x)}{\partial x} = f(t,x), \quad 0 < x < \pi, \quad 0 < t < 1,$$
$$f(t,x) = \left(2t - 6t^2 + 4t^3\right)\sin(x) - t^2(1-t)^2(2+\cos(x))\sin(x) - t^2(1-t)^2\cos(x)\sin(x),$$
$$u(0,x) = u(1,x), \quad 0 \le x \le \pi,$$
$$u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le 1, \tag{3.1}$$

for the reverse parabolic equation. It is easy to see that $u(t,x) = t^2(1-t)^2\ \sin(x)$ is the exact solution of (3.1).

For the approximate solution of nonlocal boundary value problem (3.1), consider the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$[0,1]_\tau \times [0,\pi]_h = \{(t,x_n) : t_k = k\tau, k = 1,\ldots, N-1, N\tau = 1, x_n = nh, n = 1,\ldots, M-1, Mh = \pi\}. \tag{3.2}$$

Applying (2.5), we get the first order of accuracy in t and the second order of accuracy in x

$$\begin{gathered}
\frac{u_n^k - u_n^{k-1}}{\tau} + (2+\cos(x_n))\frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} \\
- \sin(x_n)\frac{u_{n+1}^{k-1} - u_{n-1}^{k-1}}{2h} = f(t_k, x_n), \quad k = 1,\dots,N, \ n = 1,\dots,M-1, \\
u_0^k = u_M^k = 0, \quad k = 0,\dots,N, \\
u_n^0 = u_n^N, \quad n = 0,\dots,M,
\end{gathered} \tag{3.3}$$

for the approximate solutions of the nonlocal boundary value problem (3.1).

Note that for difference scheme (3.3), we have that

$$\left\{-(2+\cos(x_n))\frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} - \sin(x_n)\frac{u_{n+1}^{k-1} - u_{n-1}^{k-1}}{2h}\right\}_1^{M-1} = C_h^x u_{k-1}^h(x) + B_h^x u_{k-1}^h(x), \tag{3.4}$$

where

$$\begin{gathered}
C_h^x u^h(x) = \left\{-\frac{(2+\cos(x_{n+1}))((u_{n+1}-u_n)/h) - (2+\cos(x_n))((u_n - u_{n-1})/h)}{h}\right\}_1^{M-1}, \\
B_h^x u^h(x) = \left\{\frac{\cos(x_{n+1}) - \cos(x_n)}{h}\frac{u_n - u_{n-1}}{h} - \sin(x_n)\frac{u_{n+1}-u_n}{2h}\right\}_1^{M-1}.
\end{gathered} \tag{3.5}$$

It is easy to see that $C_h^x = (C_h^x)^*$ and $C_h^x \geq \delta I_h$, and

$$\left\|\left(C_h^x\right)^{-1} B_h^x\right\|_{L_{2h}\to L_{2h}} \leq M, \tag{3.6}$$

where I_h is the identity operator.

So, Theorems 2.1, 2.2, 2.4, and 2.5 are compatible for the solution of (3.3).

We can write (3.3) as in the matrix form

$$\begin{gathered}
A_n u_{n+1} + B_n u_n + C_n u_{n-1} = I\varphi_n, \quad n = 1,\dots,M-1, \\
u_0 = \vec{0}, \quad u_M = \vec{0}.
\end{gathered} \tag{3.7}$$

Here, φ_n is an $(N+1)\times 1$ column matrix, $A_n,\ B_n,\ C_n$ are $(N+1)\times(N+1)$ square matrices, $A_n = a_n R,\ C_n = c_n R$,

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ b_n & d & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_n & d & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_n & d & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & b_n & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_n & d \end{bmatrix}, \tag{3.8}$$

$$a_n = \frac{2+\cos(x_n)}{h^2} - \frac{\sin(x_n)}{2h}, \qquad b_n = -\frac{1}{\tau} - \frac{2(2+\cos(x_n))}{h^2},$$

$$c_n = \frac{2+\cos(x_n)}{h^2} + \frac{\sin(x_n)}{2h}, \quad d = \frac{1}{\tau}.$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \vdots \\ \varphi_n^N \end{bmatrix},$$

$$\varphi_n^0 = 0, \quad n = 1,\ldots,M-1,$$

$$\varphi_n^k = f(t_k, x_n), \quad n = 1,\ldots,N,\ n = 1,\ldots,M-1,$$

here and in the future I is the $(N+1)\times(N+1)$ identity matrix,

$$u_s = \begin{bmatrix} u_s^0 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1,\ n, n+1. \tag{3.9}$$

Samarskii and Nikolaev studied this type of system in [27] for difference equations. We seek the solution of (3.7) by the formula

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1,\ldots,1, \tag{3.10}$$

where $u_M = \vec{0}$, α_n $(n = 1, \dots, M-1)$ are $(N+1)\times(N+1)$ square matrices and β_n $(n = 1, \dots, M-1)$ are $(N+1)\times 1$ column matrices. For the solution of difference equation (3.7) we need to use the following formulas for α_n, β_n:

$$
\begin{gathered}
\alpha_n = -(B_n + C_n\alpha_{n-1})^{-1} A_n, \\
\beta_n = (B_n + C_n\alpha_{n-1})^{-1}\left(I\varphi_n - C_n\beta_{n-1}\right), \quad n = 2, \dots, M-1,
\end{gathered}
\tag{3.11}
$$

where α_1 is the $(N+1)\times(N+1)$ zero matrix and β_1 is the $(N+1)\times 1$ zero column vector.

Second, we consider again the nonlocal boundary value problem (3.1). Applying (2.6) and formulas:

$$
\begin{gathered}
\frac{u(x_{n+1}) - u(x_{n-1})}{2h} - u'(x_n) = O\left(h^2\right), \\
\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O\left(h^2\right), \\
\frac{u(x_{n+2}) - 4u(x_{n+1}) + 6u(x_n) - 4u(x_{n-1}) + u(x_{n-2})}{h^4} - u^{(4)}(x_n) = O\left(h^2\right), \\
\frac{2u(0) - 5u(h) + 4u(2h) - u(3h)}{h^2} - u''(0) = O\left(h^2\right), \\
\frac{2u(1) - 5u(1-h) + 4u(1-2h) - u(1-3h)}{h^2} - u''(1) = O\left(h^2\right),
\end{gathered}
\tag{3.12}
$$

we get the second order of accuracy in t and x

$$
\begin{aligned}
&\frac{u_n^k - u_n^{k-1}}{\tau} - (\sin(x_n) + \tau\sin(x_n) + \tau\sin(x_n)\cos(x_n))\frac{u_{n+1}^{k-1} - u_{n-1}^{k-1}}{2h} \\
&\quad + \left((2+\cos(x_n)) + \frac{\tau}{2}\left(6\cos(x_n) + 5\cos^2(x_n) - 2\right)\right)\frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} \\
&\quad + \tau(4\sin(x_n) + 2\sin(x_n)\cos(x_n))\frac{u_{n+2}^{k-1} - 2u_{n+1}^{k-1} + 2u_{n-1}^{k-1} - u_{n-2}^{k-1}}{2h^3} \\
&\quad - \frac{\tau}{2}(2+\cos(x_n))^2\frac{\left(u_{n+2}^{k-1} - 4u_{n+1}^{k-1} + 6u_n^{k-1} - 4u_{n-1}^{k-1} + u_{n-2}^{k-1}\right)}{h^4} = \varphi_n^k, \\
&\varphi_n^k = f\left(t_k - \frac{\tau}{2}, x_n\right) - \frac{\tau}{2}\left(\frac{1}{h^2}(2+\cos(x_n))\right. \\
&\qquad\qquad \times\left(f\left(t_k - \frac{\tau}{2}, x_{n+1}\right) - 2f\left(t_k - \frac{\tau}{2}, x_n\right) + f\left(t_k - \frac{\tau}{2}, x_{n-1}\right)\right) \\
&\qquad\qquad \left. - \frac{1}{2h}\sin(x_n)\left(f\left(t_k - \frac{\tau}{2}, x_{n+1}\right) - f\left(t_k - \frac{\tau}{2}, x_{n-1}\right)\right)\right),
\end{aligned}
$$

$$\begin{gathered}
k=1,\ldots,N,\quad n=2,\ldots,M-2,\\
u_0^k=u_M^k=0,\quad u_1^k=\frac{4}{5}u_2^k-\frac{1}{5}u_3^k,\ k=0,\ldots,N,\\
u_{M-1}^k=\frac{4}{5}u_{M-2}^k-\frac{1}{5}u_{M-3}^k,\quad k=0,\ldots,N,\\
u_n^0-u_n^N=0,\quad n=0,\ldots,M,
\end{gathered}\tag{3.13}$$

for the approximate solutions of the nonlocal boundary value problem (3.1).

We can rewrite this system in the following matrix form:

$$\begin{gathered}
A_nu_{n+2}+B_nu_{n+1}+C_nu_n+D_nu_{n-1}+E_nu_{n-2}=I\varphi_n,\quad n=2,\ldots,M-2,\\
u_0=\vec{0},\quad u_M=\vec{0},\quad u_1=\frac{4}{5}u_2-\frac{1}{5}u_3,\quad u_{M-1}=\frac{4}{5}u_{M-2}-\frac{1}{5}u_{M-3},
\end{gathered}\tag{3.14}$$

where φ_n is an $(N+1)\times 1$ column matrix, A_n, B_n, C_n, D_n, E_n are $(N+1)\times(N+1)$ square matrices

$$\varphi_n=\begin{bmatrix}\varphi_n^0\\ \vdots\\ \varphi_n^{\mathrm{N}}\end{bmatrix},\tag{3.15}$$

$A_n=v_nR$, $B_n=y_nR$,$D_n=z_nR$, $E_n=w_nR$,

$$C_n=\begin{bmatrix}1&0&0&\cdots&0&-1\\ r_n&p&0&\cdots&0&0\\ 0&r_n&p&\cdots&0&0\\ \vdots&\vdots&\vdots&\cdots&\vdots&\vdots\\ 0&0&0&\cdots&p&0\\ 0&0&0&\cdots&r_n&p\end{bmatrix},\tag{3.16}$$

where $p=1/\tau$,

$$\begin{aligned}
v_n&=\frac{\tau}{2h^3}(4\sin(x_n)+2\sin(x_n)\cos(x_n))-\frac{\tau}{2h^4}(2+\cos(x_n))^2,\\
y_n&=-\frac{1}{2h}(\sin(x_n)+\tau\sin(x_n)+\tau\sin(x_n)\cos(x_n))\\
&\quad+\frac{1}{h^2}\Big((2+\cos(x_n))+\frac{\tau}{2}\Big(6\cos(x_n)+5\cos^2(x_n)-2\Big)\Big)\\
&\quad-\frac{\tau}{h^3}(4\sin(x_n)+2\sin(x_n)\cos(x_n))+\frac{2\tau}{h^4}(2+\cos(x_n))^2,
\end{aligned}$$

$$r_n = -\frac{1}{\tau} - \frac{2}{h^2}\left((2+\cos(x_n)) + \frac{\tau}{2}\left(6\cos(x_n) + 5\cos^2(x_n) - 2\right)\right)$$
$$- \frac{3\tau}{h^4}(2+\cos(x_n))^2,$$
$$z_n = \frac{1}{2h}(\sin(x_n) + \tau\sin(x_n) + \tau\sin(x_n)\cos(x_n))$$
$$+ \frac{1}{h^2}\left((2+\cos(x_n)) + \frac{\tau}{2}\left(6\cos(x_n) + 5\cos^2(x_n) - 2\right)\right)$$
$$+ \frac{\tau}{h^3}(4\sin(x_n) + 2\sin(x_n)\cos(x_n)) + \frac{2\tau}{h^4}(2+\cos(x_n))^2,$$
$$w_n = -\frac{\tau}{2h^4}(2+\cos(x_n))^2 - \frac{\tau}{2h^3}(4\sin(x_n) + 2\sin(x_n)\cos(x_n)). \tag{3.17}$$

For the solution of the last matrix equation, we use the modified variant of Gauss elimination method. We seek a solution of the matrix equation of the matrix equation in the following form:

$$\begin{gathered} u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}u_{n+2} + \gamma_{n+1}, \quad n = M-2,\ldots,0, \\ u_M = \vec{0}, \quad D_M = (\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}, \\ u_{M-1} = D_M^{-1}\left[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}\right], \end{gathered} \tag{3.18}$$

where $\gamma_1 = \gamma_2 = \vec{0}$, $\alpha_1 = \beta_1$ are $(N+1)\times(N+1)$ zero matrices, $\alpha_2 = -4\beta_2 = (4/5)I$, and

$$\begin{gathered} \beta_{n+1} = -F_n^{-1}A_n, \\ \alpha_{n+1} = -F_n^{-1}(B_n + D_n\beta_n + E_n\alpha_{n-1}\beta_n), \\ \gamma_{n+1} = -F_n^{-1}(I\varphi_n - D_n\gamma_n - E_n\alpha_{n-1}\gamma_n - E_n\gamma_{n-1}), \\ F_n = (C_n + D_n\alpha_n + E_n\beta_{n-1} + E_n\alpha_{n-1}\alpha_n), \quad n = 2,\ldots,M-2. \end{gathered} \tag{3.19}$$

Now, let us give the results of the numerical analysis. In order to get the solution, we used MATLAB programs. The numerical solutions are recorded for different values of $N = M$ and u_n^k represents the numerical solutions of these difference schemes at (t_k, x_n). For their comparison, the errors are computed by

$$E_M^N = \max_{-N\le k\le N, 1\le n\le M-1} \left|u(t_k, x_n) - u_n^k\right|. \tag{3.20}$$

Table 1 gives the error analysis between the exact solution and solutions derived by difference schemes. Table 1 is constructed for $N = M = 20$, 40, and 60, respectively. Hence, the second order of accuracy difference scheme is more accurate compared with the first order of accuracy difference scheme.

Table 1: Error analysis.

Difference schemes	$N = M = 20$	$N = M = 40$	$N = M = 60$
Difference scheme (3.3)	5.693×10^{-3}	2.865×10^{-3}	1.914×10^{-3}
Difference scheme (3.13)	2.390×10^{-4}	6.408×10^{-5}	2.901×10^{-5}

Table 1 is the error analysis between the exact solution and solutions derived by difference schemes.

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